STOCHASTIC PROGRAMMING APPLICATIONS

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Abstract

- In this tutorial, several recommendations for developers of stochastic programming (SP) applications are introduced.

  It is assumed that they may help people who want to apply their stochastic programming knowledge to real world problems.

- Many examples of various application areas are presented in research papers.

  They are also different by their applicability level.

  Among them, educational exercises, illustrative instances, and research case studies can be found together with true real world applications.

- Instead of discussion on selected cases, the tutorial idea is to identify common principles of successful applications and present them in the form of straightforward (and even provocative) recommendations.

  They are based on the experience of many researchers and influenced by author's personal experience with applications in engineering.

Personal remarks

I talked about SP applications before.

Therefore, at first, I liked the offer to give a tutorial about them.

Starting my preparations months ago, I understood better and better that it is not an easy task as it originally had looked. Why?

During my preparations, I rejected to talk only about my applications although I know them very well and can (like to) talk hours about them. It would too specific to restrict a talk to them.

So, I will use them only in short for illustration.

Then, I found another idea to create a list of "all" applications. It would be very boring for any audience and you can create it by using a systematic Internet search.

In addition, such a list "for one hour" cannot refer to all applications that I consider important. Which papers to skip?

So, I will use references only to those applications that support my arguments.

The missing applications are not less important and can be found easily (see COSP www page).

In addition many applications will be presented during the next week at the conference.

At the end, I have developed the following main idea for the tutorial. During my participation on the application modelling, I have collected the list of recommendations that I am finding helpful for my further applications. So, I will try to share it with you.

I hope that such a subjective recipe from my "application cookbook" can be interesting for you although it may contain a simple and obvious advice similar to: "Do not put a cup of water into the pane with hot oil."

I have to emphasize that the list does not contain only my recommendations. Many of them are inspired or proposed by my colleagues. I owe them my thanks.

So, we may agree as follows:

For recommendations that you will not like, I am the author.

For recommendations that you will like, I will try to remember original authors under the request.

I also did not want to interfere with other tutorials. With applications, it is difficult as they need to use stochastic programming knowledge in all steps.

So, what is specific for applications? What is different from the other SP areas? Let us think together:

The user who is unexperienced in stochastic programming.

Therefore, we will focus on everything what is related to the presence of a user in SP.

Let us emphasize (joking) that the presence of the user in stochastic programming applications is very unpleasant for a stochastic programmer. It is something uncertain and unknown. The resource of noise.

How it would be nice in applications without the user!

But we need them. They pay. Sometimes.

Main Goal

To share the experience learned from stochastic programming applications in the form of clearly formulated recommendations.

Why?

To help you in avoiding errors (if any or many) of ancestors who realized applications till now.

To motivate you to do real world applications (it is about future jobs for you and us as well) because we (stochastic programming community) need more real world applications as we want to advertise SP among potential users.

In addition, we all believe that stochastic programming paradigm could be applied wider than till now.

The first advice

So, I am advertising SP applications to you now and here (or more precisely here-and-now?).

There are many application areas and many applications of different purpose. So, solving your own problem, you may learn easily about single cases – visit the COSP www pages for references (e.g., see the bibliography compiled by Maarten van der Vlerk).

In addition, you may try to utilize recommendations (presented further) to your application case. However, my first recommendation is:

Recommendation 1

Do not believe my recommendations. Think and criticize them!

Organizational remarks

- We will follow common modelling steps in the titles of sections.

  However, after quite general section titles, we emphasize stochastic programming specific features.

  SP is further emphasized by the choice of examples and references.

- All sections have a similar structure:

  At the beginning, the general section description is given.

  Then recommendations follow.

  They are explained by one or more examples.

  Sometimes references and formulas are included.

Important remark

Precise references and some author's papers mentioned within the text are available from the author under the personal request.

1 INITIAL SITUATION

Why we want to apply SP?

Theory is so nice, why we should enter in the real (dirty) life?

Let us think about the situation where a strong application need appears. There are several typical situations:

- An experienced researcher needs an application (usually simple or simplified) to illustrate some features of his theoretical result, algorithmic development, etc.
- A teacher needs motivating practical examples for his course on SP.
- A PhD student needs to solve an SP application (carefully chosen and suggested by his supervisor) as it has to be part of his thesis.

  But he solves it usually only once and then after his defense he is happy to forget it for ever. The supervisor is not so lucky, he usually cannot forget so easily.
- Somebody (a user) is coming to you and needs to solve his real world decision problem.

  At the beginning, nobody knows what he really wants. In addition, the user cannot imagine consequences of his decision yet.

Example 1 (Melt Control — the beginning)

In 1994, 2 young engineers from ŽĎAS foundry were in front the task how to introduce a new information system in the factory.

They were young and ENTHUSIASTIC. They wanted everything best for "their" factory (3 more years) in all details.

They remembered something from their university times about blending models by LP.

They came to my colleague from metallurgy department (who mentioned these LP examples before) asking whether such models can be really applied to their case. Emphasizing that it is only "the cream and cherry on the top of the cake". So, the information system could be implemented without it but it would be nice ...

Because of the long tradition of very good relations between mathematical department and engineering departments within our faculty, he asked my older colleague (statistician) for an advice and collaborator.

After my return from Norway in 1994, I was invited to collaborate.

Example 2 (Further examples)

Check the application-related chapters (Chapter 1 and the case study at the end) in the book: Birge-Louveaux: Introduction to Stochastic Programming, Wiley 1996.

Do we need to use mathematical modeling?

The question has to be seriously answered for each application. Traditional YES answers are:

- For the expensive strategical decision making realized only once (e.g., do we need to build a dam?).
- For everyday's tactical decision making saving cents every minute (e.g., optimize the charge for melt control).

Recommendation 2

When building a mathematical model think about efficiency.

Example 3

However, we may give the small counterexample:

Why my e-mail addresses are deleted on the first page?

It is because the author exists in uncertain environment at home, remember the pictures (cf. with Philpott's yacht race uncertain environment).

It looks as a project management case. Unfortunately, the author is somehow responsible for this ... Stochastic parameters are almost visible. Have we tried to apply SP approach?

NO.

Why? Time savings for moving 2 floors, 40 persons, from/to 60 rooms, dealing with hundred pieces of furniture must be significant ...

NO COMMENT (inefficient to do it).

Uninformed user

There is a problem and at least two persons are involved: the user and modeler. The user does not know that he needs stochastic programming. Knowing it, he could escape immediately. So:

Recommendation 3

$\boxed{\text{Do not tell him that his problem needs stochastic programming!}}$

In the case you will tell him, he will not know what are you talking about but he definitely knows how to use Internet and finding all these ξ symbols (see previous tutorials) he could escape again (although later).

Do not talk about stochastic programming before the problem analysis (although you (as somebody who really loves stochastic programming) ALREADY KNOW that it WILL BE the SP application).

Example 4 (Warning by an example)

Once I did not follow this recommendation and my colleague was coming next day with Vajda's book on Probabilistic programming (1970) in his hands saying: "Do you want from me to accept THIS? We have much more variables than problems solved here."

It has taken some time to get him back.

Happy end: He appreciated small examples from Vajda's book later.

2 PROBLEM FORMULATION

Mathematics as a common language?

Usually, you begin your collaboration with users asking them to specify a goal.

All of them almost immediately answer: the global optimal solution and nicely smile at you as they shown that they remember something from their university studies of Calculus. They assume that because of this, I as a mathematician must be happy as they specified their goal using my language. However:

Recommendation 4

Do not trust a user that he uses your mathematical language.

This recommendation may save time of both, the stochastic program developer and user. The user uses his personal understanding of mathematical symbols. It is nice from him but it may complicate the situation.

Relation modeler/user

It does not mean that we think that users are silly. The opposite is true. Respect each other as you are partners (no opponents) and experts in different areas.

Recommendation 5

Listen and respect each other.

Common language choice

The experience tells that when the discussion between the user and stochastic program developer begins, the first important problem is to find a common language.

Recommendation 6

Find a suitable application-related common language.

Example 5 (Melt Control)

To formulate the problem in all details, step-by-step we used:

- Basic initial tool:

  Verbal description of the problem.

- Metallurgy-related tools:

  Their metallurgical schemes: They were updated (approximately 30 main schemes).

  Melt reports: Tables filled by melters.

- Model-related tools:

  Pictorial notation unifying schemes.

  Vector-based compact notation (understood as a sequence of columns describing a flow of amounts of elements through the modelled process).

  Large tables describing technological matrices and their relations.

The first choice

It has been found that verbal and mathematical descriptions have to be supported by graphical schemes. For instance, melt control model development had started with several technological schemes that were detailed by tens of metallurgical schemes. Then these schemes have been analyzed to find their common features. The conclusion was partially surprising. All known melt control technologies may be specified by relatively simple graphical schemes that use only few elementary symbols (see Popela 1998 for so called external and internal schemes).

Recommendation 7

Do not be shy to use pictorial notation and drawings to begin.

Develop a pictorial notation to simplify user-related communication.

Example 6 (Electricity market applications)

Drawings of utility function shapes are often used for communication between users and modelers.

Example 7 (Financial applications)

The efficient frontier shapes are often used by bankers to begin the discussion.

Example 8 (The irrigation pipe network design)

Water management-related schemes were used for the model development.

What users want

When we talk about the initial verbal description we have to begin with:

Recommendation 8

Ask the user (repeatedly) to specify the word: optimal solution.

Discuss the goal of optimization and ask the user whether a suboptimal solution may be acceptable.

You have to precisely understand what he means by "suboptimal". It must be defined by you!

Example 9 (Melt Control)

They had melt reports describing past melts. They wanted to get such an "optimal" charge that will be (much/enough) cheaper than historical one for similar melts in the future.

There is one big advantage if you are allowed to move from a global optimum search to the "enough good solution" search:

Recommendation 9

Remember: the suboptimal goal may lead to a solvable model!

Insight is very important

In addition remember that the user will also appreciate Geoffrion's idea (1976) that the principal benefit of modeling should be 'insight, not numbers' (cf. also comments by Philpott).

Recommendation 10

Sell insight as a result of your modeling from the beginning.

Example 10 ("Negative" investment)

In early nineties (Wild East ages = Tombstone in 19th century), one factory asked to help with verification their invoice payment policy (postponing of payments was the key goal). It had been realized by one retired man and they wanted to be sure that they will able to inherit his "common sense thinking". He was asked to write down his rules precisely.

Then the related IP was built, optimal policies were compared with true common sense policies. Differences indicated forgotten rules, and so gave the insight.

Final formulation

Reading papers, you cannot learn and understand how complicated was to get the solved problem formulation.

Recommendation 11

Remember: No problem formulation is the final one.

Example 11 (Pipe irrigation design)

After the half year of our intensive collaboration, water management people told us about a small additional difficulty. Water demands may vary during the day. How it was solved — easily as we were anticipating problems before and we have written the explicit nonanticipativity constraints relating scenarios to obtain the complete scenario-based two-stage SP.

User's viewpoints and preferences

Resigning and deciding that the last achieved problem formulation is the best one, sit down and think about user's viewpoints and preferences. Think about validity of the following sentences:

- Engineers (often) do not like randomness in the optimization models (and in ODE and PDE as well). Small examples: The discussion about melt control EV model extension; the discussion about plate heat exchanger design.
- Business people (often) do not like nonlinear terms in their models (rate related terms are considered as exceptions).
- Programmers are (often) able to use heuristic algorithms and approaches even for classical problems.

Problems and some "final" formulations

In process equipment design, the total equipment cost is optimized, see Popela (2000). The program is nonlinear and exact values of certain input parameters may be only obtained later during the construction process and related measurements. When the cost must be estimated in advance, WS approach has to be used (see one-stage programs).

In water management, the optimum reconstruction of an irrigation pipe network is based on the solution of a program combining network flow ideas with nonlinear integer terms, see Popela (1999). In this case, uncertain water demands are taken into account and recourse payment (see two-stage programs) is included when demands are not supplied.

In metallurgy, the melt control program was developed with the use of a general multistage underlying program, implemented in the foundry, and further analyzed in Popela (1998).

Further applications utilized for the forthcoming recommendations are: the efficient pipe coloring, the regression-based extension of the melt control model, scenario identification and reduction for melt control problems, continuous casting modelling, the foundry payment policy verification, Schmidt's engine control, plate heat exchanger design, pollution problems modelling, flood hedging models, marble tiles optimal production, thesis assignment, voice analysis and result reduction, financial resources allocation, and feasible time tables search.

3 PROBLEM ANALYSIS

Learn from ancestors When you are discussing with the user how to model his problem, remember that you are not the first one applying stochastic programming:

Recommendation 12

$\boxed{\text{Follow the experience of your ancestors!}}$

Example 12 Do not follow my bad example: I have learned about my ancestor in melt control modeling (Evers 1967) when my model was already built. In such situations you have to discuss pros and cons of your and ancestor's model.

Application areas

So check the COSP www pages and you will get many references as stochastic programming has been applied in the following areas:

- Agriculture
- Capacity planning
- Energy
- Engineering
- Finance
- Fisheries management
- Forestry
- Military
- Production control
- Scheduling
- Sport
- Telecommunications
- Transportation
- Water management

The book on stochastic programming applications edited mainly by W. Ziemba will appear soon.

For the classical collection see the last part of Ermoliev and Wets book from 1989 (mainly the chapter written by King).

Improvement of existing models

There are some areas that are flooded by papers on applications. For these areas you have to check ancestors and with high probability you will find a suitable SP model. Then be creative and extend previous models. The most popular areas seem to be:

- Finance with highly increased popularity since the beginning of nineties (search the books edited by Ziemba but cf. papers in Dempster ed. 1980!?)
- Water management was very popular in eighties (IIASA models see www archive; look in the collection by Ermoliev and Wets from 1988; check examples in the book by Prekopa, Stochastic programming, Kluwer, 1995;)
- Energy problems are very popular since late nineties (check the chapter written by Wallace and Fleten for Handbook of Stochastic Programming, 2004, eds. Ruszczynski and Shapiro).
- Logistics was popular before but because of SIP enhances it is an application area for a new century (check the chapter written by Powell and Topaloglu for Handbook of Stochastic Programming, 2004, eds. Ruszczynski and Shapiro but cf. the original Dantzig's example?)

Recommendation 13

Find the related conference talks and visit them.

Example 13 *(Melt Control)*

Every model is possible to improve, e.g., Evers' one stage recourse model was for a cupola furnace, my multistage one is for a general furnace.

The possibilities for new models

If you search the area for new applications, look at engineering problems. The engineering areas are promising as there are less application than in the other areas.

There are still ancestors but not so many (for systematic results check Marti's www page and contents of Springer Lecture Notes related to Munich (Germany) regular workshops).

My own experience is that stochastic programming as a branch of mathematical programming is not often implemented even in the cases that are ready for its use.

Although uncertain parameters of the considered problems are often observed and identified, an expected value approach (EV) is mostly preferred instead of more sophisticated ideas.

In these cases, either the problem complexity is not taken into account or the model developer thinks that the development of stochastic programming ended only with its remarkable theoretical achievements in the seventies.

Therefore, stochastic programs are not utilized because they are often marked as difficult to understand and their results as impenetrable.

If only deterministic models are used in engineering applications then inadequate simplified conclusions are made.

Recommendation 14

Try applications in engineering areas.

Strategical and tactical decisions

Many applications are mainly strategical (i.e. useful once) e.g. "optimal crop organization in Tanzania" (see e.g., GAMS model library for examples of such models). Every consideration has an exception: The exception here is finance were many models are for everyday use.

Recommendation 15

Solve problems for everyday use!

Till now they are less popular in applications, so they are available.

Automatic control and decision support

Usually there are two possibilities how to implement the obtained optimal solution:

- Automatic control: When it is implemented it is difficult to stop it and correct it. It is a nightmare of modelers.
- Decision support: During the implementation process, the responsible user may stop its implementation having some doubts.

Recommendation 16

Prefer the decision support approach when it is possible.

Questionnaire

Classify the problem by its verbal description by using the questionnaire similar to the next one:

1. How many decision makers participate in decision problem? (i.e. check the applicability of theory of games)
2. How many criteria have to be considered? (i.e. check the usefulness of multicriteria optimization)
3. Are decisions spatially or time distributed? (i.e. are multistage or other dynamic models useful?)
4. Any uncertain parameters have to be taken into account? (e.g., qualitative, interval, random, or fuzzy parameters)
5. Which basic types for decision variables must be used? (e.g., scalar, vectors, series, or functions)
6. Which domains are specified for decision variables? (e.g., real, integer, zero-one values)
7. Do we have to differentiate between global and local optimum? (i.e., check convexity properties)
8. May differentiability of involved functions be employed? (i.e. select smooth or non-smooth optimization techniques)
9. Are there linear functions used for problem description?
10. Is it possible to take advantage from the special structure? (e.g., network flow problems)

Recommendation 17

Apply the questionnaire to your application.

These questions form the classification scheme that helps to analyze the problem. However, to obtain a stochastic program there must be the answer ‘yes’ to question 4. The model developer should remember that many engineers among users often tend to underestimate the role of randomness at their applications.

Example 14 It is true for melt control, water management, and heat plate exchanger design as well. They belong to stochastic programming.

4 UNDERLYING PROGRAM BUILDING

Descriptive approach

The next step is to build a model formally, syntactically correctly, in a descriptive way. We do not think about semantics yet.

This step is based on traditional OR modeling schemes.

The use of the descriptive approach is advantageous even when we interpret it as the description of a set of problems that are still waiting for the future solution techniques.

Neighbourhood of the modeled problem

The general formal description also allows us to identify relations to other parts of modelled reality. Think about the model context whether it can be separated from the surrounding elements and whether the proposed optimization is not contradictory.

Recommendation 18

Is your "local" optimization also the "global" one?

Underlying program As a result of traditional OR model building, we obtain a mathematical program, however it also involves random parameters:

$$? \in \underset{\mathbf{x}}{\mathrm{argmin}}\{f(\mathbf{x};\boldsymbol{\xi}) \mid \mathbf{x} \in C(\boldsymbol{\xi}) = \{\mathbf{x} \in X \mid \mathbf{g}(\mathbf{x};\boldsymbol{\xi}) \circ \mathbf{0}\}\}, \quad (1)$$

where $\mathbf{x}$ denotes a decision, $X \subset \mathbb{R}^n$, $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\mathbf{g} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. Our task is to find at least one optimal solution $\mathbf{x}_{\min}$, from the feasible set C, minimizing the objective $f(\mathbf{x})$. The symbol argmin denotes a set of all optimal solutions – minimizers. The question mark ? together with $\in$ is used to distinguish (1) for instance from the program, where all optimal solutions are searched. The symbol $\circ$ represents a column vector of symbols $\circ_i \in \{\le, \ge, =\}$, and the feasible set may be often rewritten as $C(\boldsymbol{\xi}) = \bigcap_{i=1}^{m} C_i(\boldsymbol{\xi}) = \bigcap_{i=1}^{m}\{\mathbf{x} \in X \mid g_i(\mathbf{x},\boldsymbol{\xi}) \circ_i 0\}$ or $C(\boldsymbol{\xi}) = \{\mathbf{x} \in X \mid g_i(\mathbf{x},\boldsymbol{\xi}) \le 0, 1 \le i \le l; g_i(\mathbf{x},\boldsymbol{\xi}) = 0, l+1 \le i \le m\}$. Since the program (1) is derived from a mathematical program resulting from the replacement of some constants by random parameters, it is called the underlying mathematical program (UP) (e.g., Prekopa 95).

Meaning of the underlying program What is the meaning of the given program (1)? It is quite clear after observation $\boldsymbol{\xi}^s$ substitutes for $\boldsymbol{\xi}$, but what happens to the program (1), when the realization of the randomness is not observed? Although the description (1) is unclear, it helps from the modelling point of view. It is understood as a syntactically correct description, for which semantics will be given later.

Recommendation 19

$\boxed{\text{Obtain UP by OR modeling + identification of randomness.}}$

The most general model

As the UP is meaningless, it is not a bad idea to build it as general as it is possible (see ideas of the coming conference lecture by Pennanen).

Recommendation 20

Build the most general UP.

The general framework main disadvantage is that we may remain too far from the useful applications and their solution. However, when we simplify we do it under the precise control.

Modeling shortcut

Engineering applications are often modeled using the I/O systems. Such a description identifies input and output variables, and their connection explicitly by a vector function.

In such situations we have the main part of deterministic OR modeling done. Although the general description of the optimization model may have a non-explicit form (e.g., computational software may represent the function) the forthcoming discussion is also useful for this case.

Typically, the inverse problem is solved: Find the input parameter values for which the given output vector is obtained.

It may have an infinite number of feasible solutions. Then, the optimization program is derived by the choice of one output as the objective function value and by the assignment of bounds to other outputs.

If the model parameters are named then some of them may be considered as uncertain. There is the challenge what to do next.

OR reminder

We are not repeating model building related comments included in the first tutorial as the USER is under focus.

Recommendation 21

$\boxed{\text{Use analogy!}}$

Example 15

My melt control model building was inspired mainly by similarities to blending problems in chemical engineering and "investment mix" in finance.

Recommendation 22

$\boxed{\text{However, analogy cannot hide differences!}}$

Example 16

Compare simple recourse cases in finance with incomplete recourse cases in melt control.

Difficulties in identification of random parameters

Related problems were discussed during other tutorials (Philpott and Pflug). There are still many assumptions usually required in SPs that do not apply in certain cases.

Recommendation 23

$\boxed{\text{Be careful about decision dependent randomness!}}$

Example 17

Soros as a big investor; dominant raw material in melt control.

What to do when it appears: Check the conference program; search Jonsbraaten PhD thesis.

In general what is the role of assumptions in applications do we need to stop when they are not satisfied?

Recommendation 24

$\boxed{\text{Try to simplify as suboptimal might be enough good.}}$

What about another models?

There is a competition of modelers. There is no guarantee that your SP approach will be chosen.

Recommendation 25

Your arguments must be ready for competition of modelers.

Example 18 (Melt Control) Who are the competitors in Melt Control?

- Regression analysis (does not improve the past)
- Control to standard (does not improve the past)
- Optimal control based on PDE models (not enough input data)

We have to analyze problems and advertise the SP way of thinking.

5 DETERMINISTIC REFORMULATION

Feasibility

$$
\begin{aligned}
C &= C(\boldsymbol{\xi}) \ \text{ for } \ \mathbf{g}(\mathbf{x};\boldsymbol{\xi}) \equiv \mathbf{g}(\mathbf{x}). \\
C^{\text{IS}} &= \{\mathbf{x} \in X \mid \mathbf{x} \in C(\boldsymbol{\xi}^s)\} \ \text{ for fixed } s. \\
C^{\text{EV}} &= \{\mathbf{x} \in X \mid \mathbf{x} \in C(E\boldsymbol{\xi})\} \\
C^{\text{EC}} &= \{\mathbf{x} \in X \mid \mathbf{x} \in E_\xi C(\boldsymbol{\xi})\} \\
C^{\text{WS}} &= \{\mathbf{x}(\boldsymbol{\xi}) \in X \mid \mathbf{x}(\boldsymbol{\xi}) \in C(\boldsymbol{\xi}) \ \text{ a.s.(almost surely)}\} \\
C^{\text{AS}} &= \{\mathbf{x} \in X \mid \mathbf{x} \in \bigcap_{\xi^s \in \Xi} C(\boldsymbol{\xi}^s)\} \\
C^{\text{AS}} &= \{\mathbf{x} \in X \mid \mathbf{x} \in C(\boldsymbol{\xi}) \ \text{ a.s.}\} \\
C^{\text{SB}} &= \{\mathbf{x} \in X \mid \mathbf{x} \in \bigcap_{\xi \in \Xi^s \subset \Xi} C(\boldsymbol{\xi})\}, \\
C^{\text{JP}} &= \{\mathbf{x} \in X \mid \mathcal{P}(\mathbf{x} \in C(\boldsymbol{\xi})) \geq \alpha\}, \\
C^{\text{JP}} &= \bigcap_{k \in K} \{\mathbf{x} \in X \mid \mathcal{P}(\mathbf{x} \in \bigcap_{i \in I_k} C_i(\boldsymbol{\xi})) \geq \alpha_k\} \\
& \quad\ \{I_k\}_{k=1}^{K} \subset 2^I \\
C^{\text{SP}} &= \{\mathbf{x} \in X \mid \mathcal{P}(\mathbf{x} \in C_i(\boldsymbol{\xi})) \geq \alpha_i, i = 1, \ldots, m\}, \\
C^{\text{RP}} &= \{\mathbf{x} \in X \mid \exists \mathbf{y}(\boldsymbol{\xi}) \in Y : \mathbf{h}(\mathbf{g}(\mathbf{x};\boldsymbol{\xi}), \mathbf{y}(\boldsymbol{\xi})) \circ \mathbf{0}, \ \text{ a.s.}\}
\end{aligned}
$$

Recommendation 26

$\boxed{\text{Specify feasibility.}}$

Optimality

$$
\begin{aligned}
z = f(\mathbf{x}) &= f(\mathbf{x};\boldsymbol{\xi}). \\
z^{\mathrm{IS}} = f^{\mathrm{IS}}(\mathbf{x}) &= f(\mathbf{x};\boldsymbol{\xi}^s) \text{ for fixed } s. \\
z^{\mathrm{EV}} = f^{\mathrm{EV}}(\mathbf{x}) &= f(\mathbf{x};E\boldsymbol{\xi}). \\
z^{\mathrm{MM}} = f^{\mathrm{MM}}(\mathbf{x}) &= \sup_{\xi\in\Xi} f(\mathbf{x};\boldsymbol{\xi}). \\
z^{\mathrm{WS}}(\boldsymbol{\xi}) &= f^{\mathrm{WS}}(\mathbf{x}(\boldsymbol{\xi});\boldsymbol{\xi}) = f(\mathbf{x};\boldsymbol{\xi}). \\
z^{\mathrm{DP}} &= E_\xi\{\min_{\mathbf{x}(\xi)} f^{\mathrm{WS}}(\mathbf{x}(\boldsymbol{\xi});\boldsymbol{\xi})\}. \\
\mathbf{x}^{\mathrm{AV}} &= E_\xi \mathbf{x}(\boldsymbol{\xi}). \\
z^{\mathrm{HN}} = f^{\mathrm{HN}}(\mathbf{x}) &= \mathcal{E}_\xi f(\mathbf{x};\boldsymbol{\xi}). \\
z^{\mathrm{EO}} = f^{\mathrm{EO}}(\mathbf{x}) &= E_\xi f(\mathbf{x};\boldsymbol{\xi}). \\
z^{\mathrm{SB}} = f^{\mathrm{SB}}(\mathbf{x}) &= \sum_{s\in\mathcal{S}} p_s f(\mathbf{x};\boldsymbol{\xi}^s). \\
z^{\mathrm{CO}} = f^{\mathrm{CO}}(\mathbf{x}(\boldsymbol{\xi}_1)) &= \mathcal{E}_{\xi_2|\xi_1} f(\mathbf{x}(\boldsymbol{\xi}_1);\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) \\
& E_{\xi_1} \min_{\mathbf{x}(\xi_1)} E_{\xi_2|\xi_1} f(\mathbf{x}(\boldsymbol{\xi}_1);\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) \\
z^{\mathrm{MV}} = f^{\mathrm{MV}}(\mathbf{x}) &= E_\xi f(\mathbf{x};\boldsymbol{\xi}) + \lambda \mathrm{var}_\xi f(\mathbf{x};\boldsymbol{\xi}) \\
z^{\mathrm{MV}} = f^{\mathrm{MV}}(\mathbf{x}) &= E_\xi f(\mathbf{x};\boldsymbol{\xi}) + \lambda \sqrt{\mathrm{var}_\xi f(\mathbf{x};\boldsymbol{\xi})} \\
z^{\mathrm{PO}} = f^{\mathrm{PO}}(\mathbf{x}) &= \mathcal{P}(f(\mathbf{x};\boldsymbol{\xi}) \geq c) \\
z^{\mathrm{QO}} = f^{\mathrm{QO}}(\mathbf{x}) &= \inf_c\{c \mid \mathcal{P}(f(\mathbf{x};\boldsymbol{\xi}) \geq c) \leq \alpha\} \\
z^{\mathrm{RP}} = f^{\mathrm{RP}}(\mathbf{x}) &= E_\xi\{f(\mathbf{x};\boldsymbol{\xi}) + Q(\mathbf{x};\boldsymbol{\xi})\} \\
z^{\mathrm{GU}} = f^{\mathrm{GU}}(\mathbf{x}) &= \mathcal{U}(f(\mathbf{x};\boldsymbol{\xi}))
\end{aligned}
$$

Recommendation 27

$\boxed{\text{Specify optimality.}}$

Successful couples

$$\min_{\mathbf{x}} \{ f^{\text{IS}}(\mathbf{x}) \mid \mathbf{x} \in C^{\text{IS}} \}$$

$$\min_{\mathbf{x}(\xi)} \{ f^{\text{WS}}(\mathbf{x}(\boldsymbol{\xi}); \boldsymbol{\xi}) \mid \mathbf{x}(\xi) \in C^{\text{WS}}$$

$$E_{\xi} \{ \min_{\mathbf{x}(\xi)} \{ f(\mathbf{x}(\boldsymbol{\xi}); \boldsymbol{\xi}) \mid \mathbf{x}(\boldsymbol{\xi}) \in X, \mathbf{g}(\mathbf{x}(\boldsymbol{\xi}); \boldsymbol{\xi}) \circ \mathbf{0} \text{ a.s.} \} \}$$

$$\sum_{s \in \mathcal{S}} p_s (\min_{\mathbf{x}^s} \{ f(\mathbf{x}^s; \boldsymbol{\xi}^s) \mid \mathbf{x}^s \in X, \mathbf{g}(\mathbf{x}^s; \boldsymbol{\xi}^s) \circ \mathbf{0} \})$$

$$\min_{\mathbf{x}} \{ f^{\text{EV}}(\mathbf{x}) \mid \mathbf{x} \in C^{\text{EV}} \}$$

$$\min_{\mathbf{x}} \{ f^{\text{MM}}(\mathbf{x}) \mid \mathbf{x} \in C^{\text{AS}} \}$$

$$\min_{\mathbf{x}} \{ \sup_{s \in \mathcal{S}} f(\mathbf{x}; \boldsymbol{\xi}^s)) \mid \mathbf{x} \in X, \mathbf{g}(\mathbf{x}; \boldsymbol{\xi}^s) \circ \mathbf{0}, s \ \in \mathcal{S} \}$$

$$\min_{\mathbf{x}} \{ f^{\text{HN}}(\mathbf{x}) \mid \mathbf{x} \in C^{\text{AS}} \}$$

$$\min_{\mathbf{x}} \{ f^{\text{SB}}(\mathbf{x}) \mid \mathbf{x} \in C^{\text{SB}} \}$$

$$\min_{\mathbf{x}} \{ f^{\text{GU}}(\mathbf{x}) \mid \mathbf{x} \in C \cap C^{\text{JP}} \cap C^{\text{SP}} \}$$

$$\min_{\mathbf{x}} \{ f^{\text{EO}}(\mathbf{x}) \mid \mathbf{x} \in C^{\text{EC}} \}$$

Recommendation 28

$\boxed{\text{Feasibility + Optimality = Deterministic Reformulation.}}$

Be careful, users often prefer simply interpretable formulations (EV) and results.

TS models:

$$\min_{\mathbf{x},\mathbf{y}}\{f(\mathbf{x};\boldsymbol{\xi})+q(\mathbf{x};\mathbf{y};\boldsymbol{\xi})\mid$$
$$\mathbf{x}\in C_x,\mathbf{h}(\mathbf{g}(\mathbf{x};\boldsymbol{\xi});\mathbf{y})\circ\mathbf{0},\mathbf{y}\in Y\}$$

$$\min_{\mathbf{x}}\{E_\xi\{f(\mathbf{x};\boldsymbol{\xi})+\min_{\mathbf{y}(\xi)}q(\mathbf{x};\mathbf{y}(\boldsymbol{\xi});\boldsymbol{\xi})\}\mid$$
$$\mathbf{x}\in C_x,\mathbf{h}(\mathbf{g}(\mathbf{x};\boldsymbol{\xi});\mathbf{y}(\boldsymbol{\xi}))\circ\mathbf{0},\mathbf{y}(\boldsymbol{\xi})\in Y,\text{a.s.}\}$$

$$\min_{\mathbf{x},\mathbf{y}}\{\mathbf{c}^\top\mathbf{x}+\mathbf{q}^\top(\boldsymbol{\xi})\mathbf{y}\mid\mathbf{A}\mathbf{x}=\mathbf{b},\mathbf{x}\geq\mathbf{0},$$
$$\mathbf{T}(\boldsymbol{\xi})\mathbf{x}+\mathbf{W}(\boldsymbol{\xi})\mathbf{y}=\mathbf{h}(\boldsymbol{\xi}),\mathbf{y}\geq\mathbf{0}\}$$
$$\min_{\mathbf{x}}\{E_\xi\{\mathbf{c}^\top\mathbf{x}+Q(\mathbf{x};\boldsymbol{\xi})\}\mid$$
$$\mathbf{x}\in C_x=\{\mathbf{x}\geq\mathbf{0}\mid\mathbf{A}\mathbf{x}=\mathbf{b}\}\}$$
$$Q(\mathbf{x};\boldsymbol{\xi})=\inf_{\mathbf{y}(\xi)}\{\mathbf{q}(\boldsymbol{\xi})^\top\mathbf{y}(\boldsymbol{\xi})\mid$$
$$\mathbf{y}(\boldsymbol{\xi})\in C_y(\mathbf{x};\boldsymbol{\xi})=\{\mathbf{y}(\boldsymbol{\xi})\geq\mathbf{0}\mid$$
$$\mathbf{T}(\boldsymbol{\xi})\mathbf{x}+\mathbf{W}(\boldsymbol{\xi})\mathbf{y}(\boldsymbol{\xi})=\mathbf{h}(\boldsymbol{\xi})\,\text{a.s.}\}\}$$

$$\min_{\mathbf{x}}\{f(\mathbf{x})+\sum_{s=1}^{S}p_sQ(\mathbf{x};\boldsymbol{\xi}^s)\mid\mathbf{x}\in C_x\}$$
$$Q(\mathbf{x};\boldsymbol{\xi}^s)=\min_{\mathbf{y}_s}\{q(\mathbf{x};\mathbf{y}_s;\boldsymbol{\xi}^s)\mid\mathbf{y}_s\in C_y(\mathbf{x};\boldsymbol{\xi}^s)\}$$
$$\min_{\mathbf{x}}\{\mathbf{c}^\top\mathbf{x}+\sum_{s=1}^{S}p_sQ(\mathbf{x};\boldsymbol{\xi}^s)\mid\mathbf{A}\mathbf{x}=\mathbf{b},\mathbf{x}\geq\mathbf{0}\}$$
$$Q(\mathbf{x};\boldsymbol{\xi}^s)=\min_{\mathbf{y}_s}\{\mathbf{q}_s^\top\mathbf{y}_s\mid\mathbf{W}_s\mathbf{y}_s=\mathbf{h}_s-\mathbf{T}_s\mathbf{x},\mathbf{y}_s\geq\mathbf{0}\}$$
$$\min_{\mathbf{x},\mathbf{y}_s:s\in\mathcal{S}}\{f(\mathbf{x})+\sum_{s=1}^{S}p_sq(\mathbf{x};\mathbf{y}_s;\boldsymbol{\xi}^s)\mid$$
$$\mathbf{x}\in C_x,\mathbf{y}_s\in C_y(\mathbf{x};\boldsymbol{\xi}^s),s\in\mathcal{S}\}$$
$$\min_{\mathbf{x},\mathbf{y}_s:s\in\mathcal{S}}\{\mathbf{c}^\top\mathbf{x}+\sum_{s=1}^{S}p_s\mathbf{q}_s^\top\mathbf{y}_s\mid$$
$$\mathbf{A}\mathbf{x}=\mathbf{b},\mathbf{x}\geq\mathbf{0},\mathbf{W}_s\mathbf{y}_s=\mathbf{h}_s-\mathbf{T}_s\mathbf{x},\mathbf{y}_s\geq\mathbf{0},s\in\mathcal{S}\}$$

$$
\begin{array}{lllll}
\min_{\mathbf{x},\mathbf{y}_s:s\in\mathcal{S}} \mathbf{c}^\top\mathbf{x} + \sum_{s=1}^{S} p_s\mathbf{q}_s^\top\mathbf{y}_s & & & & \\
\mathbf{A}\mathbf{x} & & & = & \mathbf{b} \\
\mathbf{T}_1\mathbf{x} & + & \mathbf{W}_1\mathbf{y}_1 & = & \mathbf{h}_1 \\
\vdots & & \ddots & & \vdots \\
\mathbf{T}_S\mathbf{x} & + & \quad\quad \mathbf{W}_S\mathbf{y}_S & = & \mathbf{h}_S \\
\mathbf{x} \geq \mathbf{0} & & \mathbf{y}_1 \geq \mathbf{0} \ \ldots \ \mathbf{y}_S \geq \mathbf{0}. & &
\end{array}
$$

$$
\min_{\mathbf{x}_s,\mathbf{y}_s:s\in\mathcal{S}} \{\sum_{s=1}^{S} p_s(f(\mathbf{x}_s) + q(\mathbf{x}_s;\mathbf{y}_s;\boldsymbol{\xi}^s)) \mid
$$

$$
\mathbf{x}_s \in C_x, \mathbf{y}_s \in C_y(\mathbf{x}_s;\boldsymbol{\xi}^s), s \in \mathcal{S}, \forall r,u \in \mathcal{S} : \mathbf{x}_r = \mathbf{x}_u\}
$$

$$
\min_{\mathbf{x}_s,\mathbf{y}_s:s\in\mathcal{S}} \{\sum_{s=1}^{S} p_s(\mathbf{c}^\top\mathbf{x}_s + \mathbf{q}_s^\top\mathbf{y}_s) \mid
$$

$$
\mathbf{A}\mathbf{x}_s = \mathbf{b}, \mathbf{x}_s \geq \mathbf{0},
$$

$$
\mathbf{W}_s\mathbf{y}_s = \mathbf{h}_s - \mathbf{T}_s\mathbf{x}_s, \mathbf{y}_s \geq \mathbf{0}, s \in \mathcal{S}, \forall r,u \in \mathcal{S} : \mathbf{x}_r = \mathbf{x}_u\}
$$

$$
\begin{array}{lllll}
\min_{\mathbf{x}_s,\mathbf{y}_s:s\in\mathcal{S}} \sum_{s=1}^{S} p_s(\mathbf{c}^\top\mathbf{x}_s + \mathbf{q}_s^\top\mathbf{y}_s) & & & & \\
\mathbf{A}\mathbf{x}_s & & & = & \mathbf{b} \\
\mathbf{T}_s\mathbf{x}_s & + & \mathbf{W}_s\mathbf{y}_s & = & \mathbf{h}_s \\
\mathbf{x}_r - \mathbf{x}_u & & & = & \mathbf{0} \\
\mathbf{x}_r, \mathbf{x}_u, \mathbf{x}_s \geq \mathbf{0} & & \mathbf{y}_s \geq \mathbf{0}. & & \\
s, r, u = 1, \ldots, S & & & &
\end{array}
$$

Strong need of simple examples:

See examples from all other tutorial. Many small examples are in the B-L book.

Recommendation 29

$\boxed{\text{Simple examples are necessary for understanding of users.}}$

Examples:

$$\max\{\mathbf{c}^\top \mathbf{x} \mid \mathbf{a}^\top \mathbf{x} \leq b(\xi), \mathbf{x} \in [0;1]^n\}.$$

$$\begin{gathered}
\max\{10x_1 + 15x_2 + 20x_3\} \\
5x_1 + 10x_2 + 20x_3 \leq \xi \\
x_1, x_2, x_3 \in [0;1], \\
p_1 = P(\xi = \xi^1) = P(\xi = 3) = 0,2 \\
p_2 = P(\xi = \xi^2) = P(\xi = 12) = 0,3 \\
p_3 = P(\xi = \xi^3) = P(\xi = 25) = 0,5
\end{gathered}$$

WS model:

$$\max\{\ \mathbf{c}^\top \mathbf{x}(\xi) \mid \mathbf{a}^\top \mathbf{x}(\xi) \leq b(\xi), \mathbf{x} \in [0;1]^n\}$$

$$\begin{gathered}
\max\{\ \mathbf{c}^\top \mathbf{x}(\xi^1) \mid \mathbf{a}^\top \mathbf{x}(\xi^1) \leq 3, \mathbf{x}(\xi^1) \in [0;1]^3\} \\
\max\{\ \mathbf{c}^\top \mathbf{x}(\xi^2) \mid \mathbf{a}^\top \mathbf{x}(\xi^2) \leq 12, \mathbf{x}(\xi^2) \in [0;1]^3\} \\
\max\{\ \mathbf{c}^\top \mathbf{x}(\xi^3) \mid \mathbf{a}^\top \mathbf{x}(\xi^3) \leq 25, \mathbf{x}(\xi^3) \in [0;1]^3\}
\end{gathered}$$

$$\begin{gathered}
\mathbf{x}^{\mathrm{ws}}(\xi^1) = (0,6;0;0)^\top,\ z^{\mathrm{ws}}(\xi^1) = 6 \\
\mathbf{x}^{\mathrm{ws}}(\xi^2) = (1;0,7;0)^\top,\ z^{\mathrm{ws}}(\xi^2) = 20,5 \\
\mathbf{x}^{\mathrm{ws}}(\xi^3) = (1;1;0,5)^\top,\ z^{\mathrm{ws}}(\xi^3) = 35 \\
z^{\mathrm{ws}} = E_\xi z^{\mathrm{ws}}(\xi) = \textstyle\sum_{i=1}^3 p_i z^{\mathrm{ws}}(\xi^i) = 24,85
\end{gathered}$$

MM model:

$$\max\{\ \mathbf{c}^\top\mathbf{x} \mid \mathbf{a}^\top\mathbf{x} \le \inf_{\xi\in\Xi} b(\xi), \mathbf{x} \in [0;1]^n\}$$
$$\max\{(10;15;20)\mathbf{x} \mid (5;10;20)\mathbf{x} \le 3, \mathbf{x} \in [0;1]^3\}$$
$$\mathbf{x}^{\text{MM}} = (0,6;0;0)^\top, z^{\text{MM}} = 6$$

EV model:

$$\max\{\ \mathbf{c}^\top\mathbf{x} \mid \mathbf{a}^\top\mathbf{x} \le E_\xi b(\xi), \mathbf{x} \in [0;1]^n\}$$
$$\max\{(10;15;20)\mathbf{x} \mid (5;10;20)\mathbf{x} \le 16,7, \mathbf{x} \in [0;1]^3\}$$
$$\mathbf{x}^{\text{EV}} = (1;1;0,085)^\top, z^{\text{EV}} = 26,7$$

PC model:

$$\max\{\ \mathbf{c}^\top\mathbf{x} \mid P(\mathbf{a}^\top\mathbf{x} \le b(\xi)) \ge \alpha, \mathbf{x} \in [0;1]^n\}.$$
$$F(t) = P(b(\xi) < t), F^{-1}(\alpha) = \inf\{t \mid F(t) \ge \alpha\}$$
$$\max\{\ \mathbf{c}^\top\mathbf{x} \mid \mathbf{a}^\top\mathbf{x} \le F^{-1}(1-\alpha), \mathbf{x} \in [0;1]^n\}\alpha = 0,7$$
$$\max\{(10;15;20)\mathbf{x} \mid (5;10;20)\mathbf{x} \le 10, \mathbf{x} \in [0;1]^3\}$$
$$\mathbf{x}^{\text{PC}} = (1;0,7;0)^\top, z^{\text{PC}} = 20,5$$

TS + TS EV model:

$$\max\{\mathbf{c}^\top\mathbf{x} - q(\xi)y^-(\xi) \mid$$
$$\mathbf{a}^\top\mathbf{x} + y^+(\xi) - y^-(\xi) = b(\xi), \mathbf{x} \in [0;1]^n\},$$

$$\max\{\mathbf{c}^\top\mathbf{x} - [E_\xi q(\xi)]y^- \mid$$
$$\mathbf{a}^\top\mathbf{x} + y^+ - y^- = E_\xi b(\xi), \mathbf{x} \in [0;1]^n\}$$

$$\text{TS WS, MM, EV} \sim \text{OS WS, MM, EV}$$
$$q(\xi^1) = 2, q(\xi^2) = 3, q(\xi^3) = 6, \text{TS PC } z^{\text{PC}} = 16,9$$

TS SB model:

$$\max\{\mathbf{c}^\top\mathbf{x} - \textstyle\sum_{i=1}^{3} p_i q(\xi^i)y^-(\xi^i) \mid$$
$$\mathbf{a}^\top\mathbf{x} + y^+(\xi^i) - y^-(\xi^i) = b(\xi^i), \forall i, \mathbf{x} \in [0;1]^n\}$$
$$\mathbf{x} = (1;1;0)^\top, y^-(\xi^1) = 12,$$
$$y^-(\xi^2) = 3, y^+(\xi^3) = 10, z^{\text{HN}} = 17,5$$

EVPI + VSS:

$$\text{EEV} = E_\xi\{c^\top\mathbf{x}^{\text{EV}} - q(\xi)(\mathbf{a}^\top\mathbf{x}^{\text{EV}} - b(\xi))_+\} = 16,99$$
$$z^{\text{MM}} < \text{EEV} < z^{\text{HN}} < z^{\text{WS}} < z^{\text{EV}}$$
$$\text{EVPI} = z^{\text{WS}} - z^{\text{HN}} = 7,35$$
$$\text{VSS} = z^{\text{HN}} - \text{EEV} = 0,51$$