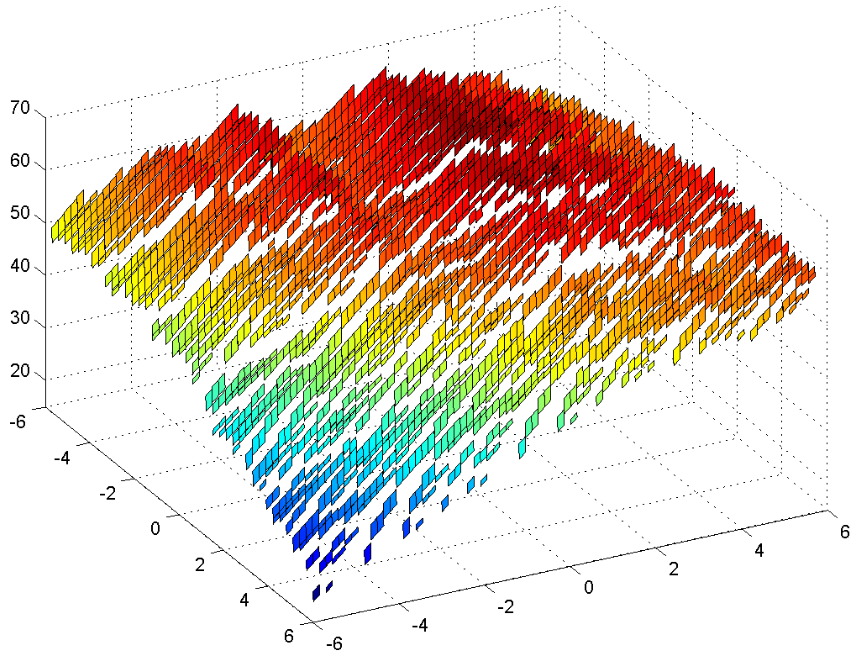
Stochastic Integer Programming

Tutorial, SPXI, Vienna, August 26, 2007



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Stochastic Integer Programming:
an introduction to future success stories?

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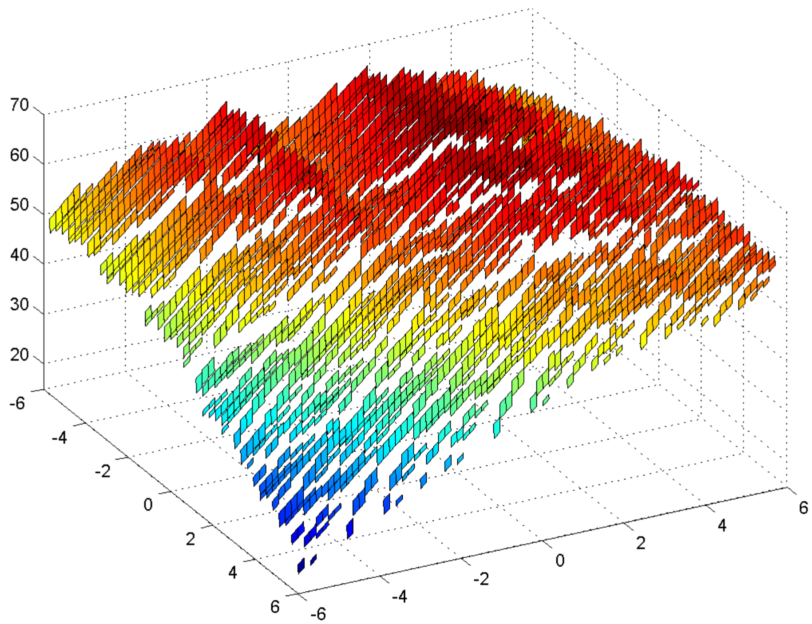

- some ‘success stories’
- lots of challenges!

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Outline

Mixed-integer recourse models

- introduction & motivation
- Simple Integer Recourse: definition, properties
- convex approximations SIR: Modification of Recourse Data
- same approach: Complete IR

Algorithms:

(a) based on structural properties

- MRD
- Structured Enumeration
- Decomposition based Branch & Bound

(b) inspired by continuous SLP and/or deterministic MIP

- large-scale MIP
- Integer L-shaped
- Stochastic Branch & Bound
- Dual Decomposition
- Cutting planes: sequential set convexification
- Branch-and-Fix Coordination

Two-stage mixed-integer recourse

Generalization of continuous recourse model:

$$\min_{x \in X} \{cx + Q(x) : Ax = b\}$$

where

$$Q(x) = \mathbb{E}_\omega \left[v\left(h(\omega) - T(\omega)x\right)\right], \qquad v(s) = \min_{y \in Y} \{qy : Wy = s\}$$

with $X = \mathbb{Z}_+^{\bar{n}} \times \mathbb{R}_+^{n-\bar{n}}$, $\quad Y = \mathbb{Z}_+^{\bar{p}} \times \mathbb{R}_+^{p-\bar{p}}$ (canonical form)

Terminology

- Q expected value function (EVF)
- v second-stage value function
- W recourse matrix

$\bar{n} = \bar{p} = 0$: continuous SLP $\qquad$ $\bar{n} > 0, \bar{p} = 0$: MIP with convex objective

Mean-risk integer models: semi-plenary Schultz (Tuesday)

Why include integer variables?

- natural integrality of decision variables
- yes/no, on/off decisions $\longrightarrow$ $0/1$ variables
- artificial indicator variables for conditional linear constraints (LP formulation of CO problems)

$$0 \leq x \leq Mz, \quad x \in \mathbb{R}, \quad z \in \{0,1\}$$

- satisfy k out of n constraints, e.g. discrete CC (semi-plenary Ahmed, Mon)

$$\Pr\{Tx \geq \omega\} \geq \alpha \in (0,1) \text{ with } \Pr\{\omega = \omega^s\} = p^s,\ s = 1,\ldots,S$$

Why not?

- continuous SLP is already difficult enough
- complexity: 2nd-stage problems NP-hard

Theoretical argument: continuous SLP is #P complete [Dyer & Stougie '06]
$\longrightarrow$ SMIP not harder (. . .)

Solution approaches for deterministic MIP:

- LP + rounding: NO GOOD
- Branch & Bound with LP relaxation
- Polyhedral theory: valid inequalities
- Lagrangian relaxation
- Benders' decomposition

Combine with SLP algorithms $\longrightarrow$ algorithms for SMIP?   2nd part

First: SMIP is a battle between

- stochastics: GOOD
- integrality: BAD

Usually, result is UGLY: non-convex, . . .

Sometimes result is BEAUTIFUL: convex!

SMIP applications

machine investment	Rinnooy Kan, Stougie (1988)
sequencing	Jørnsten (1992)
scheduling	Birge, Dempster (1996), Tayur, Thomas, Natraj (1995)
routing	Laporte, Louveaux, Mercure (1992), Verweij et al. (2003)
location	Laporte, Louveaux, Van Hamme (2002)
unit commitment	Takriti, Birge, Long (1996), Carøe, Ruszczynski, Schultz (1997)
	Philpott, Römisch, Dentcheva, Schultz, Escudero ea, . . .
electricity distribution	Klein Haneveld, VdV (2000)
pollution control	Ruszczynski, Ermoliev, Norkin (1995)
ALM	Dert (1995), Drijver, KH, VdV (2003), Streutker, KH, VdV (2006)
stochastic GAP	Albareda, VdV, Fernandez (2006)
network expansion	Norkin ea (1995)
interdiction	Morton, Pan (2004)
design	Crainic, Lium, Wallace (2004)
natural gas value chain opt.	Tomasgard, Fodstad (2004)
paratransit service	Higle (2004), Cremers, KH, VdV (2006)
supply chain planning	Alonso-Ayuso et al. (2003)

See S(I)P Bibliography (2003) and SP E-Print Series via `http://stoprog.org`

Simple Integer Recourse

By definition, the second-stage problem is

$$v(s) := \min_{y^+,y^-} \{q^+ y^+ + q^- y^- : \quad \begin{aligned} & y^+ \geq s \\ & y^- \geq -s \\ & y^+ \in \mathbb{Z}_+^m,\ y^- \in \mathbb{Z}_+^m \ \} \end{aligned}$$

with $q^+ \geq 0$, $q^- \geq 0$ $\quad \longrightarrow v > -\infty$: sufficiently expensive

Special case: $T(\omega) = T$ fixed, $h(\omega) = \omega$

SR structure $\longrightarrow$ v and Q separable in tender variables $s = Tx$

$$v(s) \quad = \quad \sum_{i=1}^{m} \left(q_i^+ \lceil s_i \rceil^+ + q_i^- \lfloor s_i \rfloor^- \right), \qquad s \in \mathbb{R}^m$$

where $\lceil t \rceil^+ := \max\{0, \lceil t \rceil\}$ and $\lfloor t \rfloor^- := \max\{0, -\lfloor t \rfloor\}$

Q also separable

$$Q(s) = \sum_{i=1}^{m} \mathbb{E}_{\omega_i}\left[q_i^+\lceil \omega_i - s_i \rceil^+ + q_i^- \lfloor \omega_i - s_i \rfloor^-\right], \qquad s \in \mathbb{R}^m$$

Consider one-dimensional generic SIR functions

$$v(s) = q^+\lceil s \rceil^+ + q^- \lfloor s \rfloor^-, \qquad s \in \mathbb{R}$$

$$Q(x) = q^+\mathbb{E}_\omega\left[\lceil \omega - x \rceil^+\right] + q^-\mathbb{E}_\omega\left[\lfloor \omega - x \rfloor^-\right], \qquad x \in \mathbb{R}$$

with $\omega \in \Omega \subset \mathbb{R}$

Compare to continuous SR analogues

$$\bar{v}(s) = q^+(s)^+ + q^-(s)^-$$

$$\bar{Q}(x) = q^+\mathbb{E}_\omega\left[(\omega - x)^+\right] + q^-\mathbb{E}_\omega\left[(\omega - x)^-\right]$$

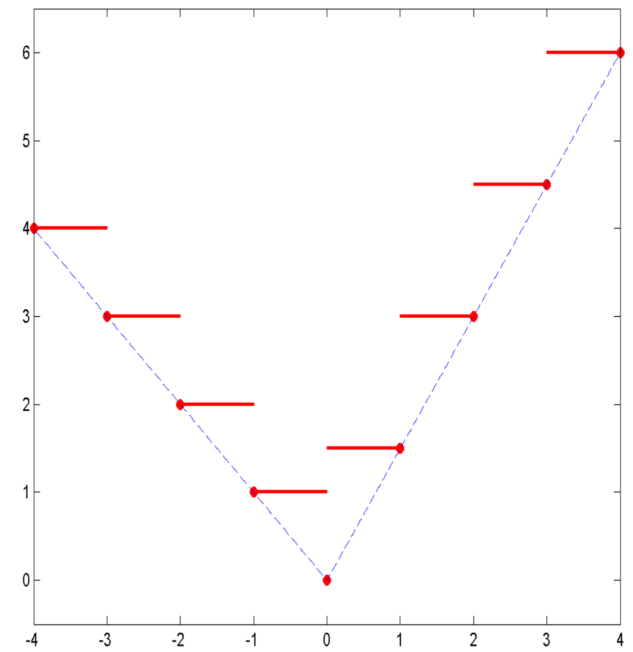
Same, except for rounding

Properties of

$$v(\omega - x) = q^+\lceil \omega - x\rceil^+ + q^-\lfloor \omega - x\rfloor^-,\ x \in \mathbb{R}$$

with ω fixed

- discontinuous at $x = \omega + k,\ k \in \mathbb{Z}$
  jump q^+ at $x = \omega + k,\ k \in \mathbb{Z}_+$
  jump q^- at $x = \omega - k,\ k \in \mathbb{Z}_+$
- right continuous at $x > \omega$
  left continuous at $x < \omega$
  neither at $x = \omega$
- piecewise constant
- lower semicontinuous: $v(s) \leq \lim_{u \longrightarrow s} v(u)$
- non-convex



v and $\bar{v}$ (dashed), $q^+ = 1$, $q^- = 1.5$

One-dimensional generic SIR function Q [Louveaux & VdV '93, VdV '95]

Useful formula: for $x \in \mathbb{R}$

$$\begin{aligned} Q(x) &= q^+ \mathbb{E}_\omega \left[\lceil \omega - x \rceil^+ \right] + q^- \mathbb{E}_\omega \left[\lfloor \omega - x \rfloor^- \right] \\ &= q^+ \sum_{k=0}^{\infty} \Pr\{\omega > x + k\} + q^- \sum_{k=0}^{\infty} \Pr\{\omega < x - k\} \end{aligned}$$

Compare to continuous SR analogue $\bar{Q}$:

$$\begin{aligned} \bar{Q}(x) &= q^+ \mathbb{E}_\omega \left[(\omega - x)^+ \right] + q^- \mathbb{E}_\omega \left[(\omega - x)^- \right] \\ &= q^+ \int_x^{\infty} \Pr\{\omega > s\} ds + q^- \int_{-\infty}^{x} \Pr\{\omega < s\} ds \end{aligned}$$

Properties of Q

Finite iff $\mathbb{E}_\omega[|\omega|] < +\infty$ (assume from now on)

Continuity

Recall: for ω fixed

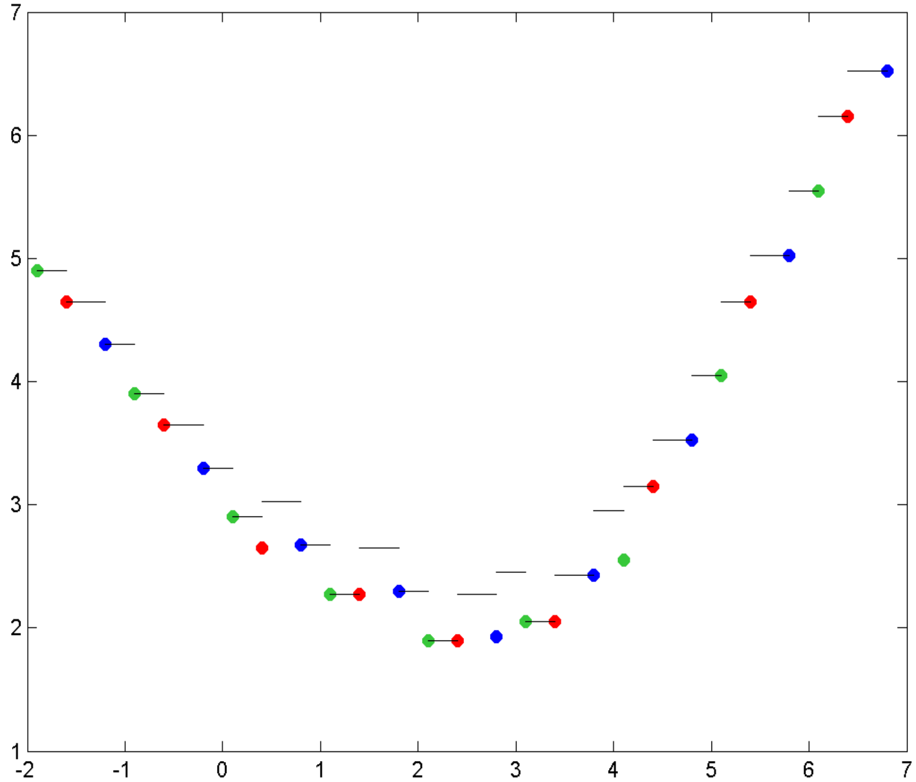
- v is lower semicontinuous
- $v(\omega - x)$ is discontinuous at $x = \omega + k, \; k \in \mathbb{Z}$
- constant in between

Q is lower semicontinuous for any distribution of ω

Q is discontinuous at x if and only if $\Pr\{\omega \in x + \mathbb{Z}\} > 0$

If ω is discretely distributed with support Ω then

- Q is discontinuous at $\Omega + \mathbb{Z}$
- constant in between

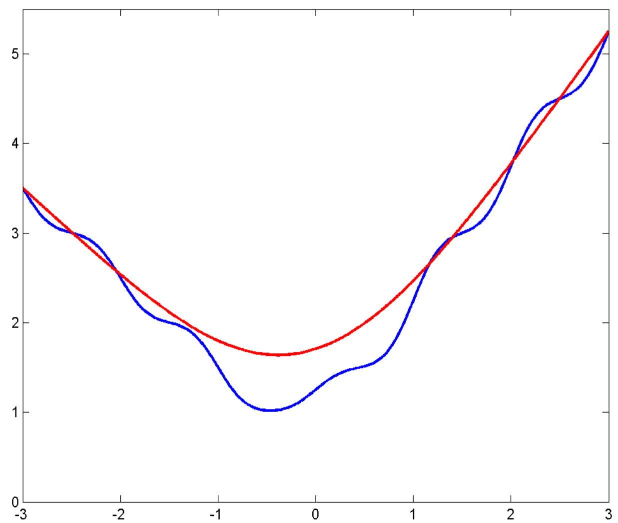


$$\Omega = \{0.4, 2.8, 4.1\} \; p = (0.25, 0.35, 0.4)$$

$$q^{+} = 1 \quad q^{-} = 1.5$$

Q is discontinuous at x if and only if $\mathrm{Pr}\{\omega \in x + \mathbb{Z}\} > 0$

$\longrightarrow$ Q is continuous if and only if ω is continuously distributed



$\omega \sim N(0, 0.05)$ $\quad q^+ = 1,\ q^- = 1.5$

$\omega \sim N(0, 1)$

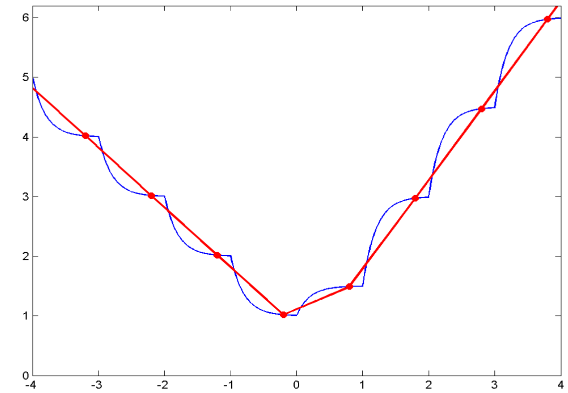
(Lipschitz continuity, differentiability)

Convexity

In general Q is non-convex: expectation of 'step function' v

For all distributions of ω,
Q convex on $\alpha + \mathbb{Z}$ $\quad \forall \alpha \in [0, 1)$

$\longrightarrow$ 'reasonable' convex approximations . . .



$\omega \sim \mathrm{Exp}(5)$, $\alpha = 0.8$

Q convex $\Rightarrow$ ω continuous, but $\not\Leftarrow$

Theorem [Klein Haneveld, Stougie, VdV '06]
SIR function Q is convex iff $\omega \sim$ pdf f with

$$f(s) = G(s + 1) - G(s), \quad s \in \mathbb{R}$$

where G is an arbitrary cdf with finite mean

Example: for any $\alpha \in [0, 1)$
G discrete on $\alpha + \mathbb{Z}$ $\longrightarrow$ f constant on $[\alpha + k, \alpha + k + 1)$, $k \in \mathbb{Z}$

Algorithms for SIR

Approach:

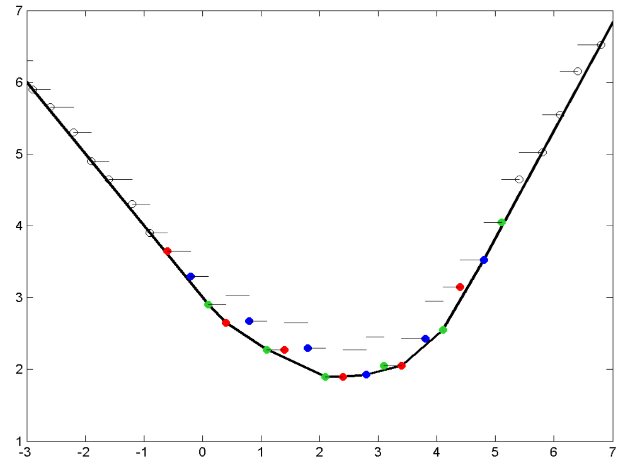
- construct convex approximation Q
- show: equivalent to Simple Continuous Recourse function $\bar{Q}_\xi$

$\longrightarrow$ approximate solution using SCR algorithm

Discrete distribution RHS ω

[Klein Haneveld, Stougie, VdV '96]

- construct convex hull of Q
- strongly polynomial algorithm
  $\longrightarrow$ ξ in SCR



Continuous distribution RHS ω: *Modification of Recourse Data*

Modification of Recourse Data

Recourse model with random RHS $\omega \sim$ cdf F

$$\begin{array}{ll} \min\limits_{x} & cx + Q(x) \\ \text{s.t.} & x \in X := \left\{x \in \mathbb{R}^{n_1}_+ : Ax = b\right\} \end{array}$$

$$Q(x) := \mathbb{E}_\omega[v(\omega - Tx)] \qquad v(s) := \begin{array}{ll} \min\limits_{y} & qy \\ \text{s.t.} & Wy = s, \quad s \in \mathbb{R}^m \\ & y \in Y \end{array}$$

with $Y \subset \mathbb{R}^n$ specifying

- variable type
- simple bounds

Recourse structure: (q, W, Y)

Recourse function $Q \sim$ recourse data (q, W, Y, F)

Difficult to solve? Depends on (q, W, Y, F)

Conceptual algorithm:

- input: $(q, W, Y, F) \to Q$
- output: $(\tilde{q}, \tilde{W}, \tilde{Y}, \tilde{F}) \to \tilde{Q}$

such that $\tilde{Q} \approx Q$ and $\min_{x \in X} cx + \tilde{Q}(x)$ 'easy' to solve

Examples:

To (approximately) solve CR

$$(q, W, \mathbb{R}^n_+, F),\ F \text{ continuous}$$

solve CR

$$(q, W, \mathbb{R}^n_+, \tilde{F}),\ \tilde{F} \text{ discrete}$$

To (approximately) solve SIR

$$\left([q^+, q^-],\ W,\ \mathbb{Z}^{2m}_+,\ F\right)$$

solve continuous SR

$$\left([q^+, q^-],\ [I, -I],\ \mathbb{R}^{2m}_+,\ \tilde{F}\right)$$

MRD also for

- Multiple S(I)R
- Mixed-integer recourse
- Complete integer recourse

1. Simple integer recourse

To show: 'solvable' as continuous SR

SR structure: v separable $\longrightarrow$ consider for $s \in \mathbb{R}$

$$v(s) \quad = \quad q^+ \lceil s \rceil^+ + q^- \lfloor s \rfloor^- \qquad (q^+, q^- \geq 0)$$

and SIR recourse function (tender $z_{\not i} = T_{\not i} x \in \mathbb{R}$)

$$Q(z) \quad := \quad \mathbb{E}_\omega[v(\omega - z)] = q^+ \mathbb{E}_\omega\big[\lceil \omega - z \rceil^+\big] + q^- \mathbb{E}_\omega\big[\lfloor \omega - z \rfloor^-\big]$$

$\omega \in \Omega \subset \mathbb{R}$ with cdf F (pdf f), finite mean μ

Recall: Q convex iff $\omega \sim$ pdf f with

$$f(s) = G(s+1) - G(s), \quad s \in \mathbb{R}$$

where G is a cdf with finite mean

In particular: for fixed $\alpha \in [0,1)$

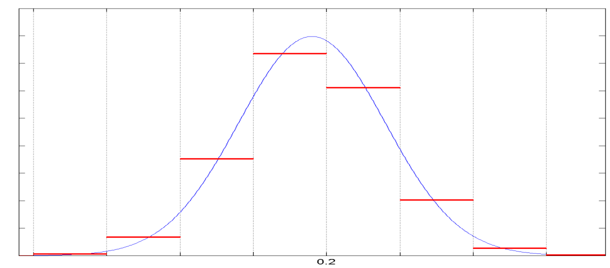
$$f \text{ constant on } [\alpha+k, \alpha+k+1), \quad k \in \mathbb{Z}$$

($G \sim$ discrete on $\alpha + \mathbb{Z}$)

Approximate $\omega \sim F$ by $\omega_\alpha \sim f_\alpha$, $\alpha \in [0,1)$

$$f_\alpha(t) := F(\lceil t \rceil_\alpha) - F(\lceil t \rceil_\alpha - 1), \quad t \in \mathbb{R}$$

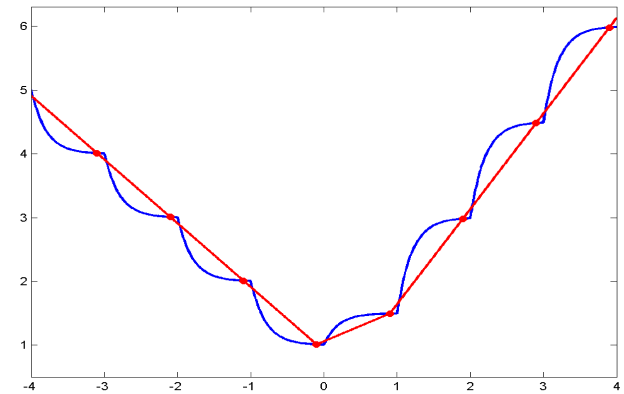
with $\lceil t \rceil_\alpha := \lceil t - \alpha \rceil + \alpha$ round up wrt $\alpha + \mathbb{Z}$



Then the *α-approximation*

$$\begin{aligned} Q_\alpha(z) \quad &:= \quad q^+ \mathbb{E}_{\omega_\alpha}\left[\lceil \omega_\alpha - z \rceil^+\right] \\ &\qquad + q^- \mathbb{E}_{\omega_\alpha}\left[\lfloor \omega_\alpha - z \rfloor^-\right] \end{aligned}$$

is a convex approximation of Q



Error bound: $\|Q_\alpha - Q\|_\infty \leq \max\{q^+, q^-\} \, \frac{TV(f)}{4}$

The convex function Q_α 'looks like' a continuous SR function . . .

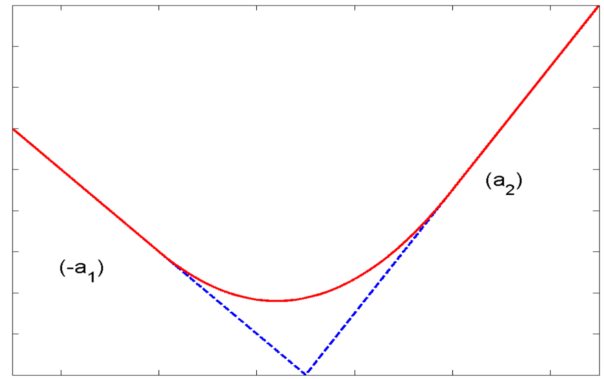
Theorem 2 [KH, S, VdV '93]

Let $\varphi(s)$, $s \in \mathbb{R}$

- convex (non-linear)
- asymptotes with slopes

$-a_1$ as $s \to -\infty$

a_2 as $s \to \infty$



Then φ is a SCR function (+ known const.):

$$\varphi(s) = a_1 \mathbb{E}_\xi\left[(\xi - s)^+\right] + a_2 \mathbb{E}_\xi\left[(\xi - s)^-\right] + C$$

where ξ is a random variable with cdf Φ

$$\Phi(t) = \frac{\varphi'_+(t) + a_1}{a_1 + a_2}$$

Apply Theorem 2 to Q_α: for $\alpha \in [0,1)$

$$Q_\alpha(z) = q^+ \mathbb{E}_{\xi_\alpha}\left[(\xi_\alpha - z)^+\right] + q^- \mathbb{E}_{\xi_\alpha}\left[(\xi_\alpha - z)^-\right] + \frac{q^+ q^-}{q^+ + q^-}$$

where ξ_α is discrete on $\alpha + \mathbb{Z}$ with cdf

$$\Phi_\alpha(t) = \frac{q^+ F(\lceil t \rceil_\alpha - 1) + q^- F(\lceil t \rceil_\alpha)}{q^+ + q^-}$$

and F is the cdf of ω

Conclusion: to (approximately) solve SIR

$$\left([q^+, q^-],\ W_{\mathsf{SIR}},\ \mathbb{Z}_+^{2m},\ F\right)$$

solve continuous SR

$$\left([q^+, q^-],\ [I, -I],\ \mathbb{R}_+^{2m},\ \Phi_\alpha\right)$$

Observe: Φ_α discrete distribution

2. Complete integer recourse

To show: 'solvable' as continuous CR

Assumptions:

- recourse complete & sufficiently expensive $\longrightarrow$ v finite
- $\mathbb{E}_\omega[||\omega||]$ finite $\longrightarrow$ Q finite

Integrality $\longrightarrow$ v and Q non-convex in x

Computation $Q(x)$: solve IP for each $\omega \in \Omega$

Special case: W is Totally Unimodular

$$\min_y \left\{qy : Wy \geq s,\ y \in \mathbb{Z}_+^n\right\} \qquad \text{(IP}(s))$$

e.g., network flow, shortest path, . . .

Then IP(s)=LP(s) (with integer solution) if right-hand side s is integer

However, in CIR right-hand side is $\omega - Tx \in \mathbb{R}^m$ . . .

Assume W is Totally Unimodular

For $s \in \mathbb{R}^m$ (!!)

$$\begin{aligned} v(s) &= \min_y \{qy : Wy \geq s,\ y \in \mathbb{Z}^n_+\} \\ &= \min_y \{qy : Wy \geq \lceil s \rceil,\ y \in \mathbb{R}^n_+\} \\ &= \max_\lambda \{\lambda \lceil s \rceil : \lambda W \leq q,\ \lambda \in \mathbb{R}^m_+\} \end{aligned}$$

Recourse complete & suff. expensive:

$\Lambda := \{\lambda \in \mathbb{R}^m_+ : \lambda W \leq q\}$ is bounded, non-empty $\longrightarrow$

$\Lambda = \text{conv}\{\lambda^1, \ldots, \lambda^K\}$, $\quad \lambda^k \geq 0$ extreme points

Hence

$$v(s) = \max_{k \in \mathcal{K}} \lambda^k \lceil s \rceil, \quad s \in \mathbb{R}^m$$

with $\mathcal{K} := \{1, \ldots, K\}$

Consider $\lambda\lceil s \rceil$, $\lambda \in \mathbb{R}^m_+$, and for $z \in \mathbb{R}^m$

$$\mathcal{R}(z) := \lambda\, \mathbb{E}_\omega[\lceil \omega - z \rceil] = \sum_{i=1}^m \lambda_i\, \mathbb{E}_{\omega_i}[\lceil \omega_i - z_i \rceil]$$

$$= \sum_{i=1}^m \lambda_i \left(\mathbb{E}_{\omega_i}\left[\lceil \omega_i - z_i \rceil^+\right] - \mathbb{E}_{\omega_i}\left[\lfloor \omega_i - z_i + 1 \rfloor^-\right]\right) \qquad \text{(if } \omega \text{ continuous)}$$

$\longrightarrow$ properties similar to SIR function Q

In particular, $\mathcal{R}$ is non-convex in general

Convex approximations of $\mathcal{R}$: replace ω by ω_α

For $\alpha \in [0,1)^m$, define $\omega_\alpha \sim f_\alpha$ such that, for $l \in \mathbb{Z}^m$

f_α constant on C^l_α and $\Pr\{\omega_\alpha \in C^l_\alpha\} = \Pr\{\omega \in C^l_\alpha\}$

where $C^l_\alpha := \prod_{i=1}^m \left(\alpha_i + l_i - 1, \alpha_i + l_i\right]$

$\mathcal{R}_\alpha$ is α-approximation of $\mathcal{R}$

$$\mathcal{R}_\alpha(z) \quad := \quad \lambda\, \mathbb{E}_{\omega_\alpha}[\lceil \omega_\alpha - z \rceil]\,, \qquad z \in \mathbb{R}^m$$

$$\overset{\text{Thm 2}}{=} \quad \lambda\, \mathbb{E}_{\xi_\alpha}[\xi_\alpha - z] = \lambda(\mu_\alpha - z)$$

with ξ_α discrete on $\alpha + \mathbb{Z}^m$,

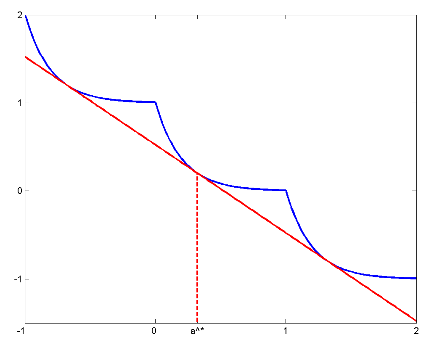
$$\Pr\{\xi_\alpha = \alpha + l\} = \Pr\{\omega \in C_\alpha^l\}, \quad l \in \mathbb{Z}^m$$

and mean value μ_α

With $\alpha^\star$ optimal choice of $\alpha \in [0,1)^m$

$\mathcal{R}_{\alpha^\star}$ is convex hull of $\mathcal{R}$

($\alpha^\star$ depends only on $F \sim \omega$)



Example: $\omega \sim \mathrm{Exp}(5) \longrightarrow \alpha^\star = 0.3219$

Recall CIR function

$$Q(z) \;:=\; \mathbb{E}_\omega\left[\min_y \left\{qy : Wy \geq \omega - z,\ y \in \mathbb{Z}^n_+\right\}\right] = \mathbb{E}_\omega\left[\max_{k\in\mathcal{K}} \lambda^k \lceil \omega - z \rceil\right]$$

Note: $Q(z) \geq \max_{k\in\mathcal{K}} \mathcal{R}^k(z), \quad \mathcal{R}^k(z) := \lambda^k \mathbb{E}_\omega[\lceil \omega - z \rceil]$

For $\alpha \in [0,1)^m$, define α-approximation of Q

$$Q_\alpha(z) := \mathbb{E}_{\omega_\alpha}\left[\min_y \left\{qy : Wy \geq \omega_\alpha - z,\ y \in \mathbb{Z}^n_+\right\}\right]$$

Theorem [VdV '04] If W is TU, then $Q_{\alpha^\star}$ is the convex hull of Q.
Moreover,

$$Q_{\alpha^\star}(z) = \mathbb{E}_{\xi_{\alpha^\star}}\left[\min_y \left\{qy : Wy \geq \xi_{\alpha^\star} - z,\ y \in \mathbb{R}^n_+\right\}\right]$$

with $\xi_{\alpha^\star} \sim$ cdf $\Phi_{\alpha^\star}$ discrete on $\alpha^\star + \mathbb{Z}^m$

$$\Pr\{\xi_{\alpha^\star} = \alpha^\star + l\} = \Pr\{\omega \in C^l_{\alpha^\star}\}, \quad l \in \mathbb{Z}^m$$

Example:

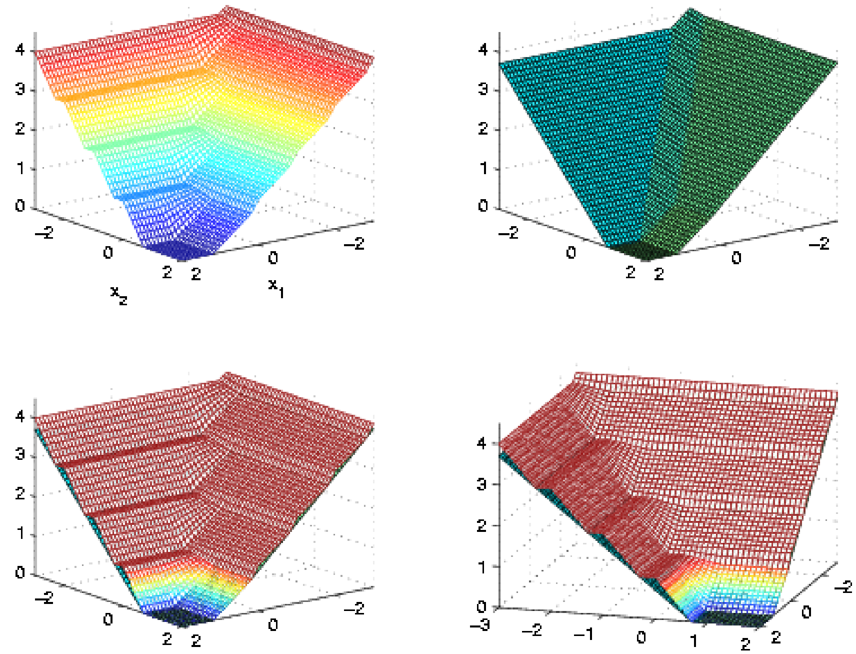
$$v(s) = \max\{\lceil s_1 \rceil, \lceil s_2 \rceil, 0\}$$

ω uniform on $(0, 0.7) \times (0, 1.2)$

$\longrightarrow \alpha^\star = (0.7,\ 0.2)$

$$\Pr\{\xi_{\alpha^\star} = (0.7, 0.2)\} = 1/6$$

$$\Pr\{\xi_{\alpha^\star} = (0.7, 1.2)\} = 5/6$$



Conclusion: to (approximately) solve TU-CIR

$$(q,\ W,\ \mathbb{Z}_+^m,\ F)$$

solve continuous CR

$$(q,\ W,\ \mathbb{R}_+^m,\ \Phi_{\alpha^\star})$$

Observe: $\Phi_{\alpha^\star}$ discrete distribution

CIR (non-TU, W integral)

$\max_{k \in \mathcal{K}} \lambda^k \lceil s \rceil \leq v(s), \; s \in \mathbb{R}^m \longrightarrow Q_{\alpha^\star}$ is a convex lower bound for Q

Alternative convex lower bound

$$Q^{\mathsf{LP}}(x) := \mathbb{E}_\omega \left[\min_y \{ qy : Wy \geq \omega - Tx, \; y \in \mathbb{R}^n_+ \} \right]$$

Theorem [VdV '04]

(i) $Q_{\alpha^\star} \geq Q^{\mathsf{LP}}$

(ii) Assume $q \geq 0$ $\quad (\Rightarrow Q(x) \geq 0)$

If ω continuous then

$$Q_{\alpha^\star}(x) > 0 \quad \Rightarrow \quad Q_{\alpha^\star}(x) > Q^{\mathsf{LP}}(x)$$

(Also for ω discrete $\sim$ technical conditions)

Thus, $Q_{\alpha^\star}$ is a better lower bound than Q^{LP}

Moreover, $Q_{\alpha^\star}$ is easier to compute:

$$Q_{\alpha^\star}(x) \quad = \quad \mathbb{E}_{\xi_{\alpha^\star}}[\ldots], \qquad \xi_{\alpha^\star} \text{ discrete}$$

$$Q^{\mathsf{LP}}(x) \quad = \quad \mathbb{E}_{\omega}[\ldots], \qquad \omega \text{ continuous} \longrightarrow m\text{-dim. integral}$$

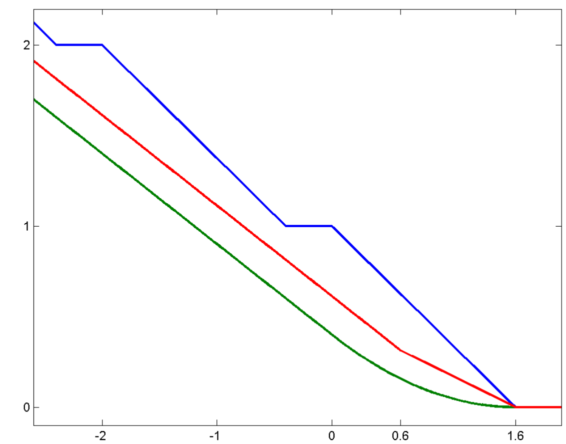
Example:

$v(s) = \min_y \{y : 2y \geq s,\ y \in \mathbb{Z}_+\}$

$\omega \sim U(0, 1.6)$

$\alpha^\star = 0.6 \qquad \Pr\{\xi_{\alpha^\star} = 0.6\} = 3/8$

$\qquad\qquad\quad \Pr\{\xi_{\alpha^\star} = 1.6\} = 5/8$



In general, approximation by $Q_{\alpha^\star}$ not good enough

Use $Q_{\alpha^\star}$ as lower bound in other algorithms, e.g.

- integer L-shaped [Laporte, Louveaux '93]
- structured enumeration [Schultz et al. '98]

Conclusion: to improve LP lower bound for CIR

$$(q,\ W,\ \mathbb{Z}^m_+,\ F)$$

solve continuous CR

$$(q,\ W,\ \mathbb{R}^m_+,\ \Phi_{\alpha^\star})$$

Observe: $\Phi_{\alpha^\star}$ discrete distribution

Outline remainder

Properties two-stage (mixed-)integer recourse model

Algorithms:

(a) based on structural properties

- (MRD)
- Structured Enumeration
- Decomposition based Branch & Bound

(b) inspired by continuous SLP and/or deterministic MIP

- large-scale MIP
- Integer L-shaped
- Stochastic Branch & Bound
- Dual Decomposition
- Cutting planes: sequential set convexification
- Branch-and-Fix Coordination

Structural properties of complete mixed-integer recourse [Schultz '93 – '98]

Assumptions

(a) Complete recourse

(b) Sufficiently expensive recourse

(c) $\mathbb{E}_\omega \big[|\omega| \big] < +\infty$

(a) + (b) $\Rightarrow$ v finite

(a) + (b) + (c) $\Rightarrow$ Q finite

Consider value function v

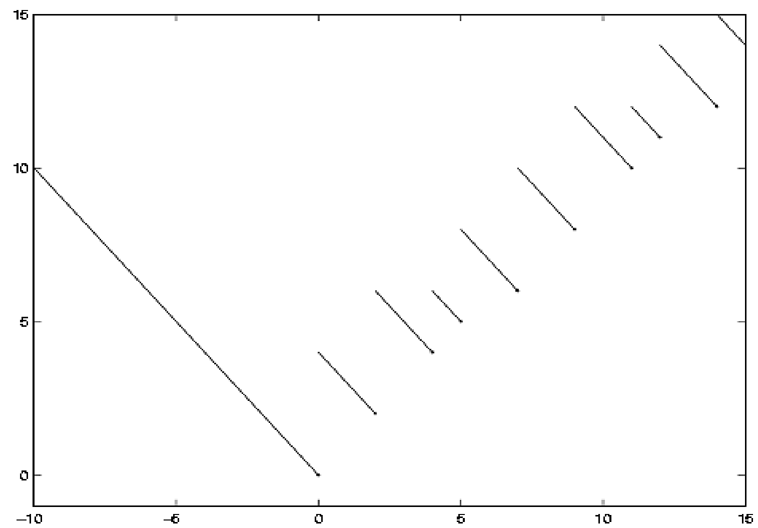
One-dimensional example

- $x \in \mathbb{R}, \quad T(\omega) = T = 1$
- $h(\omega) = \omega \in \mathbb{R}$ fixed

$$\begin{aligned} v(\omega - x) = \quad & \min \quad 2y_1 + 5y_2 + 6y_3 + y_4 \\ & \text{s.t.} \quad 2y_1 + 5y_2 + 7y_3 - y_4 = \omega - x \\ & \qquad y_1,\ y_2,\ y_3 \in \mathbb{Z}_+ \\ & \qquad y_4 \in \mathbb{R}_+ \end{aligned}$$

Properties of v

- lower semicontinuous
- possibly discontinuous at $\omega - Tx \in \mathbb{Z}$
- size of jump varies
- piecewise linear in between jumps



Continuity:

The recourse function Q is lower semicontinuous

Define, for $x \in \mathbb{R}^n$

$$D(x) = \{\omega \in \Omega : v \text{ is discontinuous at } h(\omega) - T(\omega)x\}$$

If $\Pr\{\omega \in D(\bar{x})\} = 0$ then Q continuous at $\bar{x}$

$\Pr\{\omega \in D(x)\} = 0$ for all $x \in \mathbb{R}^n$ if $h(\omega) \mid T(\omega) = T$ has a pdf for a.e. T $\longrightarrow$ Q is continuous on $\mathbb{R}^n$

Special cases:

- $h(\omega)$ and $T(\omega)$ independent, $h(\omega)$ has pdf
- $h(\omega)$ and $T(\omega)$ jointly distributed with pdf

Note: In applications, often also deterministic constraints
(e.g. flow conservation) $\longrightarrow$ Q discontinuous

Assume

- $T(\omega) = T$ fixed
- $h(\omega)$ continuous

Assume that for any nonsingular linear transformation $B : \mathbb{R}^m \mapsto \mathbb{R}^m$ each 1-dim. marginal distribution of $Bh(\omega)$ has a pdf f with $|\Delta f| < +\infty$

Then Q is Lipschitz on bounded subsets of $\mathbb{R}^n$

(Similar result for random $T(\omega)$)

Approximation of distribution $\big(T(\omega), h(\omega)\big)$ by empirical distribution

$\longrightarrow$ approximation of $Q(x^*)$ and x^* (but slowly)

Structured enumeration [Schultz, Stougie, VdV '98]

Recall: $X = \mathbb{Z}_+^{\bar{n}} \times \mathbb{R}_+^{n-\bar{n}}$, $Y = \mathbb{Z}_+^{\bar{p}} \times \mathbb{R}_+^{p-\bar{p}}$

Here $X = \mathbb{R}_+^n \cap \{x : Ax = b\}$

Assume

- $\bar{n} = 0$: continuous first stage
- $\bar{p} = p$: pure integer second stage
- only rhs $h(\omega)$ random; support $h^1, \ldots, h^S$

Basic ideas:

- structural properties of EVF Q
  $\longrightarrow$ countable (finite) set $V \subset X$, containing an optimal solution
- compute $cx + Q(x)$ for all $x \in V$, i.e.
  for every h^s, $s = 1, \ldots, S$ solve 2nd-stage IP $\min_{y \in Y}\{qy : Wy \geq Tx - h^s\}$

  Only rhs varies $\longrightarrow$ use Gröbner Basis: test set [Hemmecke & Schultz '03]

Set of candidates V

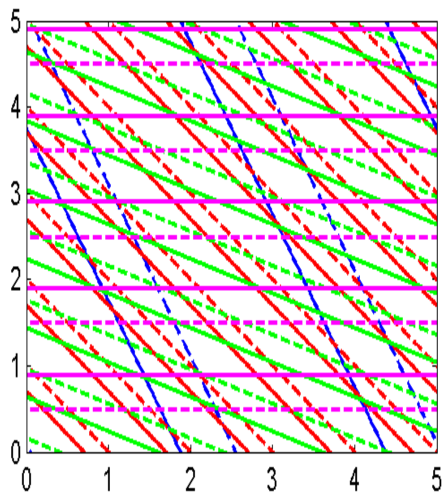
Assume W integral (rational)

$\longrightarrow \{y : Wy \geq t\} = \{y : Wy \geq \lceil t \rceil\},\ t \in \mathbb{R}^m$

$\longrightarrow v(Tx - h^s)$ is constant on $\{x : \lceil Tx - h^s \rceil = k\},\ k \in \mathbb{Z}^m$

$\longrightarrow Q$ constant on 'polyhedra': for $\bar{x} \in \mathbb{R}^n$, let $k^{(\bar{x},s)} := \lceil T\bar{x} - h^s \rceil$, $s \in S$

$$C(\bar{x}) = \bigcap_{s=1}^{S} \left\{ x \in \mathbb{R}^n : k^{(\bar{x},s)} - 1 < Tx - h^s \leq k^{(\bar{x},s)} \right\}$$



$$X = [0,5]^2 \qquad T = \begin{pmatrix} .4 & .2 \\ 1 & 1 \\ .5 & 1.25 \\ 0 & 1 \end{pmatrix}$$

$$h^s \in \left\{ \begin{pmatrix} .25 \\ .3 \\ .2 \\ .1 \end{pmatrix}, \begin{pmatrix} .98 \\ 0 \\ .8 \\ .5 \end{pmatrix} \right\}$$

DEFINITION: The countable set

$$V = \{x \in \mathbb{R}^n : x \text{ is a vertex of } C(x) \cap X\}$$

is the set of *candidate solutions*

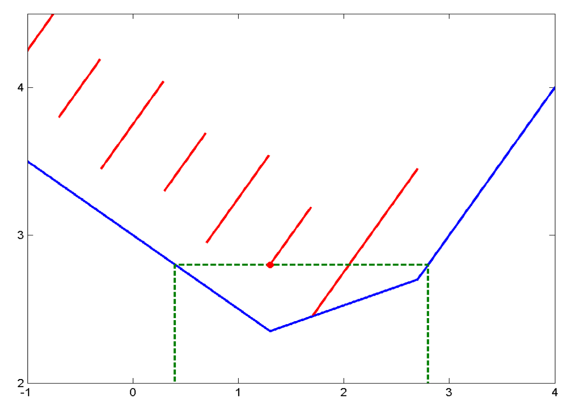
THEOREM: V contains an optimal solution

Proof: $cx + Q(x)$ is lsc, linear on each cell $C(\cdot)$

Finite subset of V: consider $Q^{\mathsf{LP}}(x) := \mathbb{E}_h\Big[\min_{y \in \mathbb{R}^p_+}\{qy : Wy \geq Tx - h\}\Big]$
Q^{LP} is convex polyhedral, and $Q^{\mathsf{LP}} \leq Q$ (alternative: Q_{α^*})

Level sets of $cx + Q^{\mathsf{LP}}$:

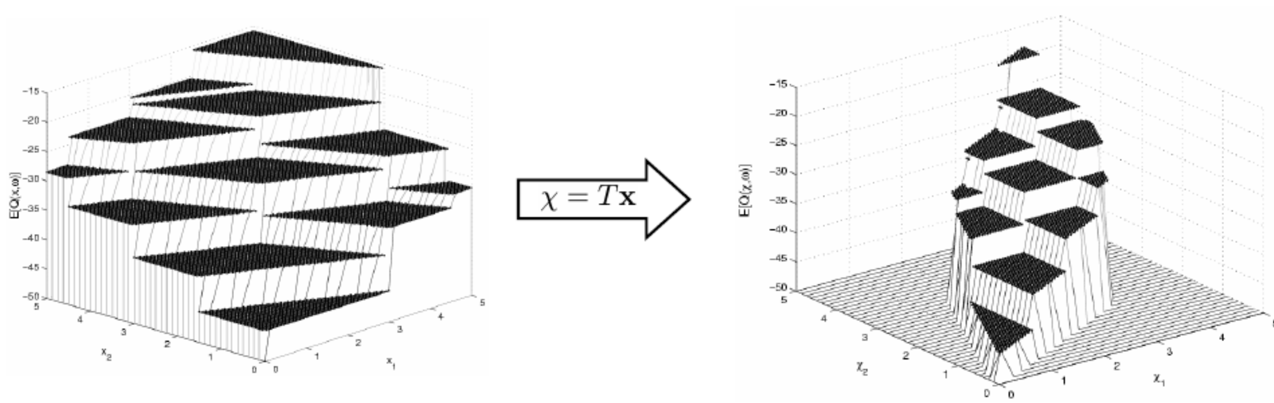
$$L(\bar{x}) := \Big\{x \in X : cx + Q^{\mathsf{LP}}(x) \leq c\bar{x} + Q(\bar{x})\Big\}$$



LEMMA: For $\bar{x} \in X$, $\underset{x \in X}{\operatorname{argmin}}\{cx + Q(x)\} \subset L(\bar{x})$

Assume $L(\cdot) \cap X$ bounded $\longrightarrow$ $V \cap L(\bar{x})$ finite

Decomposition based B&B [Ahmed, Tawarmalani, Sahinidis '04]



(picture: Shabbir Ahmed)

SMIP problem in space of tender variables:

$$\min\{f(\chi) + \sum_{s=1}^{S} p^s v(\chi - h^s) : \chi \in \mathcal{X}\}$$

with $f(\chi) := \min\{cx : x \in X,\ Tx = \chi\}$ and $\mathcal{X} := \{\chi : Tx = \chi,\ x \in X\}$

Discontinuities orthogonal to tender axes: allows to solve by B&B

- Branching: partition along discontinuities
- LB: v non-increasing and lsc
- UB: function evaluation

Large scale mixed-integer problems

MIR, discrete distrib. $\Pr\left\{\big(T(\omega), h(\omega)\big) = (T^s, h^s)\right\} = p^s,\ s = 1, \ldots, S$

$\longrightarrow$ large scale mixed-integer problem

$$\min_{x,y^s} cx + \sum_{s=1}^{S} p^s q y^s : \begin{array}{lll} Ax & & = b \\ T^s x + & W y^s & = h^s \quad \forall s \\ x \in X, & y^s \in Y\ \forall s & \end{array}$$

$$X = \mathbb{Z}_+^{\bar{n}} \times \mathbb{R}_+^{n-\bar{n}} \qquad Y = \mathbb{Z}_+^{\bar{p}} \times \mathbb{R}_+^{p-\bar{p}}$$

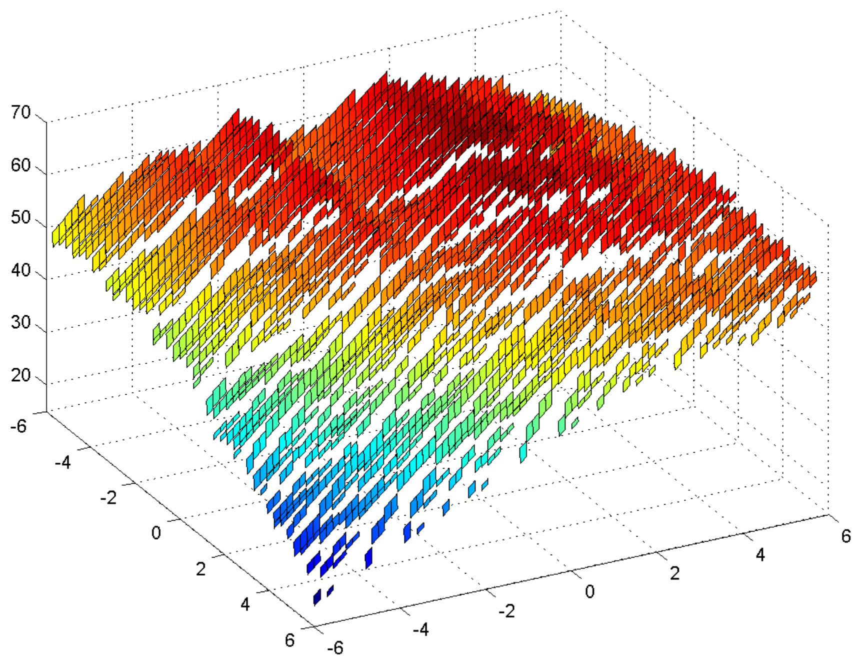
$\bar{n} + S\bar{p}$ integer variables, . . . $\longrightarrow$ For realistic values of S and $(\bar{n},)\ \bar{p}$ impossible to solve without using structure / properties

- often problem dependent
- use ideas from SLP and MIP

Stochastic multi-knapsack problem [Schultz, Stougie, VdV '98]

$$\max\left\{1.5x_1 + 4x_2 + Q(x)\ :\ x \in C = [-5,5]^2\right\}$$

$$Q(x) := \mathbb{E}_\omega[v(\omega - Tx)],\ v(s) := \begin{array}{ll} \max & 16y_1 + 19y_2 + 23y_3 + 28y_4 \\ \text{s.t.} & 2y_1 + 3y_2 + 4y_3 + 5y_4 \le s_1 \\ & 6y_1 + \ \ y_2 + 3y_3 + 2y_4 \le s_2 \\ & y_i \in \{0,1\}, i = 1,\dots,4 \end{array}$$



$$T = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$$

$\omega \sim \mathcal{U}\{5,\ 5.5,\ \dots,\ 14.5,\ 15\}^2$

$\longrightarrow$ MIP: 1764 Boolean, 882 constr.

Struc. Enum: optimal, CPU $\approx$ 0

CPLEX 5.0 – 7.1: gap 25%

CPLEX 8.1: gap 3%

Integer L-shaped algorithms

L-shaped for continuous SLP:

- complicating 1st-stage variables x: spoil separability
- for fixed x

$$Q^{\mathsf{LP}}(x) = \sum_{s=1}^{S} p^s v(\omega^s - Tx) \text{ with } v(\cdot) = \min_{y^s} \ldots$$

$\longrightarrow$ solve S deterministic LP problems (small)

Benders' decomposition for deterministic MIP:

- complicating variables $x \in \mathbb{Z}_+^p$
- fixed $x \longrightarrow$ remaining problem is LP(x)$= \min_{y \in \mathbb{R}_+^{n-p}} \ldots$

Combine L-shaped and Benders $\longrightarrow$ Integer L-shaped

L-shaped / Benders: iterations

- solve current LP problem
- generate linear optimality cuts for convex function $Q^{\text{LP}}(x)$ c.q. LP(x)

Instead of $\min\{cx + Q^{\text{LP}}(x) : x \in X\}$ solve

$$\min\{cx + \theta : \; x \in X, \; \theta \in \mathbb{R}, \; \underbrace{E_k x + \theta \geq e_k}_{\substack{\text{optimality cuts} \\ \text{derived from } Q^{\text{LP}}}}, \; k = 1, \ldots, t \; \}$$

Problem: SMIR function Q is non-convex

- approximate by linear cuts?
- how to derive?

Wollmer (MP '80)

- binary first-stage
- continuous second-stage $\longrightarrow Q(x) = Q^{\mathsf{LP}}(x)$ is convex

Laporte & Louveaux (ORL '93)

- binary first-stage
- mixed-integer second-stage
- special class of linear optimality cuts: valid only for $x \in \{0,1\}^n$

Carøe & Tind (MP '98)

- continuous first-stage
- integer second-stage (both stages mixed-integer possible)
- non-linear optimality cuts (duality theory for IP)
  $\longrightarrow$ difficult master problem

Integer L-shaped method [Laporte & Louveaux, '93]

Assumptions:

- $\bar{n} = n$: $x \in \{0,1\}^n$ essential
- $\bar{p}$ arbitrary
- $\omega \mapsto (T^s,\ h^s)$ discrete
- Q easy to compute ($Q(x)$: solve S MIP subproblems!)

Use Branch & Cut to solve MIP master

$$\min\{cx + \theta : \ x \in X,\ \theta \in \mathbb{R} \quad \underbrace{E_k x + \theta \geq e_k}_{\substack{\text{optimality cuts} \\ \text{derived from } Q}},\ k = 1, \ldots, t\ \}$$

adding linear optimality cuts at each iteration

Class of optimality cuts:

Let $x^k \in X$ be the current solution

Define the index set $I^k = \{i : x_i^k = 1\} \longrightarrow$ linear cut

$$\theta \geq \big(Q(x^k) - L\big)\left(\underbrace{\sum_{i \in I^k} x_i - \sum_{i \notin I^k} x_i}_{\leq |I^k|, \text{ with } = \text{ iff } x = x^k} - |I^k| + 1\right) + L$$

with L a global lower bound for Q

Observe: valid for $x \in \{0,1\}^n$, maybe not for $x \in (0,1)^n$

Cuts from Q^{LP} (or Q_{α^*}) can be used

Method is finite since $X \subset \{0,1\}^n$ is finite

Numerical results: 2nd stage special structure

- capacitated vehicle routing, stochastic demands [L., L., Van Hamme '02]

Stochastic Branch & Bound [Ruszczyński, Ermoliev, Norkin '98]

General setting:

$$\min\{F(x) : x \in X\}$$

with $F(x) = \mathbb{E}_\omega[f(x,\omega)]$

Applied to SMIP:

- $\bar{n}, \bar{p}$ arbitrary: 1st and 2nd stage MIP
- $\omega \mapsto (q(\omega), T(\omega), h(\omega))$ arbitrary distribution

Idea: solve by B & B (use e.g. Q^{LP} or Q_α for bounding)

Problem: repeated evaluation of $F(x) = cx + Q(x)$ too expensive

Solution: use estimates of F $\longrightarrow$ stochastic method

Details:

Iteratively

- refine partition of X
- estimate $F(x)$ on subsets
- remove subsets (. . .)

Let $P := \{X^1, \ldots, X^n\}$ be current partition of X

$$F^*(X) = \min_{X^p \in P} F^*(X^p)$$

where $F^*(U) := \min\{F(u) : u \in U\}$

Refine partition: if $x_i \in \mathbb{Z}$ then e.g. $X^j \longrightarrow X_1^j \cup X_2^j$

with $X_1^j = \{x \in X^j : x_i \leq k\}$

$X_2^j = \{x \in X^j : x_i \geq k+1\}$for a suitable $k \in \mathbb{Z}$

Computation of $F^*(X^p)$ is too expensive $\longrightarrow$ use bounds L and U

$$L(X^p) \leq F^*(X^p) \leq U(X^p), \quad X^p \in P$$

with equalities if $|X^p| = 1$

For example

$$\begin{aligned} L(X^p) &= \mathbb{E}_\omega\big[\min\{f(x,\omega) : x \in X^p\}\big] \\ U(X^p) &= \mathbb{E}_\omega[f(\bar{x},\omega)], \quad \bar{x} \in X^p \end{aligned}$$

Computation of L and U still too expensive $\longrightarrow$ use statistical estimates

$$\begin{aligned} \xi^l(X^p) &= \frac{1}{l}\sum_{i=1}^{l} \min\left\{f(x,\omega^i) : x \in X^p\right\} \\ \eta^u(X^p) &= \frac{1}{u}\sum_{i=1}^{u} f(\bar{x},\omega^i) \end{aligned}$$

with ω^i sampled from the distribution of ω

SB&B Algorithm: *Spend most time on promising subsets*

Iteratively

- Select *record set* $\bar{X} \in \underset{X^p \in P}{\operatorname{argmin}}\{\xi(X^p)\}$
- Branching: If $\bar{X}$ not a singleton $\bar{X} \longrightarrow \bar{X}_1 \cup \bar{X}_2$
- Bound estimation: update $\xi(X^p)$ and $\eta(X^p)$ for all $X^p \in P$,
  e.g. $n > 1$ observations for $\bar{X}$ c.q. $\bar{X}_1$, $\bar{X}_2$
  1 observation for other X^p
- remove X^p only if $X^p = \emptyset$
  (no bounding out, unless $\xi = L$ and $\eta = U$)

Stopping criterion: e.g. promising subsets X^p are singletons

Algorithm converges (!). Stop after n iterations: probabilistic error bound

Numerical results: several realistic problems

Dual decomposition in SMIP [Carøe & Schultz '99]

Assumptions:

- $\bar{n}, \bar{p}$ arbitrary: 1st and 2nd stage MIP
- $\omega \mapsto (T^s,\ h^s)$ discrete

Applicable to multi-stage SMIP

Idea: solve S scenario problems $\sim$ realizations $(T^s,\ h^s)$

$\longrightarrow$ solutions (x^s, y^s)

$\sim$ Lagrangian relaxation of NAC: $x^1 = x^2 = \ldots = x^S$

Problem: $x^s,\ s = 1, \ldots, S$ not equal
$\longrightarrow$ use $\bar{x} = \sum p^s x^s$, but usually $\bar{x} \notin X$ (integrality)

Solution: B & B scheme

- heuristic rounding $\bar{x}$
- use Lagrangian relaxation for bounding

Details:

Deterministic equivalent MIP

$$z = \min \left\{ cx + \sum_{s=1}^{S} p^s q y^s : (x, y^s) \in C^s \ \forall s \right\}$$

where $C^s = \left\{(x, y^s) : Ax = b, Wy^s \geq h^s - T^s x,\ x \in X,\ y^s \in Y\right\}$

Complicating variables $x \longrightarrow$ copies $x^1, \ldots, x^S$ of x

$$\min \left\{ \sum_{s=1}^{S} p^s (cx^s + qy^s) : (x^s, y^s) \in C^s \ \forall s,\ x^1 = x^2 = \ldots = x^S \right\}$$

Complicating constraints $x^1 = x^2 = \ldots = x^S \longrightarrow$ Lagrangian relaxation

Write $x^1 = x^2 = \ldots = x^S$ as $\sum_{s=1}^{S} H^s x^s = 0$

Lagrangian

$$D(\lambda) \;=\; \sum_{s=1}^{S} \min_{x^s, y^s} \left\{ p^s(cx^s + qy^s) + \lambda H^s x^s : (x^s, y^s) \in C^s \right\} =: \sum_{s=1}^{S} D^s(\lambda)$$

is separable $\longrightarrow$ S scenario subproblems

Lagrangian dual

$$z_{LD} := \max_{\lambda} D(\lambda), \qquad \lambda \in \mathbb{R}^{(S-1)n}$$

concave in λ, non-smooth $\longrightarrow$ subgradient methods

Weak duality: $z_{LD} \leq z$

Solve many times: S MIPs of size $(m_1 + m) \times (n + p)$

instead of once: single MIP of size $(m_1 + Sm) \times (n + Sp)$

Result: S scenario solutions (x^s, y^s), x^s not equal unless $z_{LD} = z$

Candidate first-stage solution: $\bar{x} := \sum p^s x^s \notin X$ (integrality)

$\longrightarrow$ use heuristic $[\bar{x}] :=$ round$(\bar{x}) \in X$

Branch & Bound scheme:

- branch on first-stage variables $\longrightarrow$ equality of x^s, $s = 1, \ldots, S$ (partition X if x_i continuous)
- use Lagrangian relaxation for bounding

Numerical results: realistic unit commitment problem, . . .

Cutting planes for SMIP: sequential set convexification [Sen & Higle '05]

Assumptions

- binary 1st stage
- 0–1 mixed-integer 2nd stage
- fixed recourse
- $\omega \mapsto (T^s, h^s)$ finite

Idea

- decomposition (stagewise): allow for many scenarios
- solve only LP relaxations of subproblems (except for UB)
- strengthen subproblem LPs sequentially $\longrightarrow$ benefit in later iterations
- utilize similarities subproblems
- pass Benders' cuts to master (solve by B&B)

Details

For fixed $\bar{x}$, SMIP decomposes in S 2nd stage 0–1 MIP problems $P(\bar{x}, \omega^s)$

Solve LP relaxation $P(\bar{x}, \omega^s) \longrightarrow y^*$ with fractional value for binary y_j^*

Generate valid inequality $\pi y \geq \pi_0$, cutting of y^*

Several ways to generate valid inequalities, e.g.

- Gomory cuts
- based on Disjunctive Decomposition [Balas '79]

$$\{y \in \mathbb{R}_+^{n_2} : Wy \geq h^s - T^s x,\ y_j \leq 0\} \cup \{y \in \mathbb{R}_+^{n_2} : Wy \geq h^s - T^s x,\ y_j \geq 1\}$$

with $y_i \leq 1$ explicitly included for all binary y_i

Derived cut $\pi y \geq \pi_0$ is valid for $P(\bar{x}, \omega^s)$

Question: ‘adapt’ coefficients $(\pi, \pi_0) \longrightarrow$ valid inequalities for all $P(x, \omega)$?

Common Cut Coefficients (C^3) Theorem: [Higle & Sen ’00]

(Conditions) There exists a function $\pi_0 : X \times \Omega \mapsto \mathbb{R}$ such that
$\pi y \geq \pi_0(x, \omega)$ is valid for $P(x, \omega)$

Problem: $\pi_0(x, \omega)$ is piecewise linear concave in x
$\longrightarrow$ convexify: linearize

- x binary: OK, since x extreme point of X
- harder in other cases ([Ntaimo & Sen ’06]: continuous 1st stage)

In iteration k, solve auxiliary LPs to obtain

- π_k, appended to W_{k-1}
- for all s: α_k^s and β_k^s ($\sim \pi_0(x, \omega)$), appended to h_{k-1}^s and T_{k-1}^s

$\longrightarrow$ valid inequalities $\pi_k y \geq \alpha_k^s - \beta_k^s x$ for $P(x, \omega^s)$

- convergence of convex hull approximations

Using updated matrices W_k, h_k^s, T_k^s, solve LP relaxations $P(\bar{x}, \omega^s)$
$\longrightarrow$ Benders’ cut to master

D^2C^3 Algorithm [Higle & Sen '05]

- binary 1st stage, 0-1 mixed 2nd stage
- Numerical results: e.g. server location [Ntaimo & Sen '04]

D^2-Branch-and-Cut [Sen & Sherali '06]: similar results

- binary 1st stage, general integer 2nd stage
- disjunction $\sim$ partial B&B tree for each subproblem

Cuts for deterministic equivalent problem [Carøe '98]

- continuous 1st stage, binary 2nd stage
- cuts in (x, y^s) space: translate to other scenarios

'Sequential pairing' for multi-stage SMIP [Guan, Ahmed & Nemhauser '06]

- all stages mixed-integer
- combine scenario problem cuts $\longrightarrow$ valid cut for tree problem
- numerical results: stochastic lot-sizing

Branch-and-Fix Coordination [Alonso-Ayuso, Escudero, Ortuño '03]

Applies to multi-stage SMIP (binary)

Key ideas:

- Lagrange relaxation NAC $\longrightarrow$ solve scenario MIP problems by B&B
- use NAC to coordinate branching

Implementation: very technical (bookkeeping)

Numerical results: realistic size, e.g. supply chain planning

Concluding remarks

SMIP has many applications

Lots of challenges!

Computational results for real-life problems

Solution methods for multi-stage models . . .

Settle for sub-optimal results

- excused by complexity: SMIP $=$ SLP $+$ MIP
- improvement over simpler models

Use of problem dependent algorithms/heuristics ($\sim$ deterministic MIP)