Formulations of Stochastic Programming Problems and Risk Aversion

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Uncertain Outcomes and Risk

Why Probabilistic Models?
- Wealth of results of probability theory
- Connection to real data via statistics
- Universal language (engineering, economics, medicine, . . .)

- Probability space \((\Omega, \mathcal{F}, P)\)
- Decision space \(\mathcal{X}\)
- Random outcome (e.g., cost) \(Z_x(\omega), \ Z : \mathcal{X} \times \Omega \rightarrow \mathbb{R}\)

Expected Value Model

\[
\min_x \mathbb{E}[Z_x] = \int_{\Omega} Z_x(\omega) \ P(d\omega)
\]

It optimizes the outcome on average (Law of Large Numbers?)

What is Risk?

Existence of unlikely and undesirable outcomes - high \(Z_x(\omega)\) for some \(\omega\)
## Classical Utility Models

### Expected Utility Models (von Neumann and Morgenstern, 1944)

\[
\min_{x \in X} \mathbb{E}\left[u(Z_x)\right] \quad \left(= \int_{\Omega} u(Z_x(\omega)) \, dP(\omega)\right)
\]

\(u : \mathbb{R} \rightarrow \mathbb{R}\) is a nondecreasing disutility function

### Rank Dependent Utility (Distortion) Models (Quiggin, 1982; Yaari, 1987)

\[
\min_{x \in X} \int_{0}^{1} F_{Z_x}^{-1}(p) \, dw(p)
\]

\(F_{Z_x}^{-1}(\cdot)\) - quantile function

\(w : [0, 1] \rightarrow \mathbb{R}\) is a nondecreasing rank dependent utility function

Existence of utility functions is derived from systems of axioms, but in practice they are difficult to elicit.
Axioms of Expected Utility Theory

\( W \) is a lottery of \( Z \) and \( V \) with probabilities \( \alpha \in (0, 1) \) and \( (1 - \alpha) \), if the probability measure \( \mu_W \) induced by \( W \) on \( \mathbb{R} \) is the corresponding convex combination of the probability measures \( \mu_Z \) and \( \mu_V \) of \( Z \) and \( V \):

\[
\mu_W = \alpha \mu_Z + (1 - \alpha) \mu_V.
\]

We write the lottery symbolically as

\[
W = \alpha Z \oplus (1 - \alpha) V.
\]

For law invariant preferences on the space of random vectors with values in \( \mathbb{R} \), von Neumann introduced the axioms:

**Independence Axiom:** For all \( Z, V, W \in \mathcal{Z} \) one has

\[
Z \prec V \implies \alpha Z \oplus (1 - \alpha) W \prec \alpha V \oplus (1 - \alpha) W, \quad \forall \alpha \in (0, 1)
\]

**Archimedean Axiom:** If \( Z \prec V \prec W \), then \( \alpha, \beta \in (0, 1) \) exist such that

\[
\alpha Z \oplus (1 - \alpha) W \prec V \prec \beta Z \oplus (1 - \beta) W
\]
Suppose the total preorder $\succeq$ on $\mathcal{Z}$ is law invariant, and satisfies the independence and Archimedean axioms. Then it has an “affine” numerical representation $U : \mathcal{Z} \to \mathbb{R}$:

$$U(\alpha Z \oplus (1 - \alpha) V) = \alpha U(Z) + (1 - \alpha) U(V).$$

If $\succeq$ is weakly continuous, then a continuous and bounded function $u : \mathbb{R} \to \mathbb{R}$ exists, such that

$$U(Z) = \mathbb{E}[u(Z)] = \int_\Omega u(Z(\omega)) \, P(d\omega).$$

New proof by separation theorem - D. & R. 2012

In a more general setting, we may consider only r.v. with finite moments, and then the boundedness condition on $u(\cdot)$ can be relaxed.
Risk-Averse Utility

\[ U(Z) = \mathbb{E}[u(Z)] = \int_{\Omega} u(Z(\omega)) \, P(d\omega) \]

Monotonicity

The total preorder \( \preceq \) is **monotonic** with respect to the partial order \( \leq \), if \( Z \leq V \implies Z \preceq V \).

We focus on \( Z \) containing integrable random vectors.

Risk Aversion

A preference relation \( \preceq \) on \( Z \) is **risk-averse**, if \( \mathbb{E}[Z|\mathcal{G}] \preceq Z \), for every \( Z \in Z \) and every \( \sigma \)-subalgebra \( \mathcal{G} \) of \( \mathcal{F} \).

Nondecreasing Convex Disutility

Suppose a total preorder \( \preceq \) on \( Z \) is weakly continuous, monotonic, risk-averse, and satisfies the independence axiom. Then the utility function \( u : \mathbb{R} \to \mathbb{R} \) is **nondecreasing and convex**.
Real random variables $Z_i$, $i = 1, \ldots, n$, are comonotonic, if

$$
(\overline{Z_i} - \underline{Z_i})(\overline{Z_j} - \underline{Z_j}) \geq 0
$$

for all $\omega, \omega' \in \Omega$ and all $i, j = 1, \ldots, n$.

**Dual Independence Axiom:** For all comonotonic random variables $Z$, $V$, and $W$ in $\mathcal{Z}$ one has

$$
Z \triangleleft V \implies \alpha Z + (1 - \alpha)W \triangleleft \alpha V + (1 - \alpha)W, \quad \forall \alpha \in (0, 1)
$$

**Dual Archimedean Axiom:** For all comonotonic random variables $Z$, $V$, and $W$ in $\mathcal{Z}$, satisfying the relations

$$
Z \triangleleft V \triangleleft W,
$$

there exist $\alpha, \beta \in (0, 1)$ such that

$$
\alpha Z + (1 - \alpha)W \triangleleft V \triangleleft \beta Z + (1 - \beta)W
$$
Affine Representation

If the total preorder $\succeq$ on $\mathcal{Z}$ is law invariant, and satisfies the dual independence and Archimedean axioms, then a numerical representation $U : \mathcal{Z} \rightarrow \mathbb{R}$ of $\succeq$ exists, which satisfies for all comonotonic $Z, V \in \mathcal{Z}$ and all $\alpha, \beta \in \mathbb{R}_+$ the equation

$$U(\alpha Z + \beta V) = \alpha U(Z) + \beta U(V).$$

Integral Representation

Suppose $\mathcal{Z}$ is the set of bounded random variables. If, additionally, $\succeq$ is continuous in $\mathcal{L}_1$ and monotonic, then a bounded, nondecreasing, and continuous function $w : [0, 1] \rightarrow \mathbb{R}_+$ exists, such that

$$U(Z) = \int_0^1 F_Z^{-1}(p) \, dw(p), \quad Z \in \mathcal{Z}.$$
Risk Averse Dual Utility

\[ U(Z) = \int_0^1 F_Z^{-1}(p) \, dw(p), \quad Z \in \mathcal{Z} \quad (*) \]

Risk Aversion

A preference relation \( \preceq \) on \( \mathcal{Z} \) is risk-averse, if \( \mathbb{E}[Z|G] \preceq Z \), for every \( Z \in \mathcal{Z} \) and every \( \sigma \)-subalgebra \( G \) of \( \mathcal{F} \).

Convex Rank-Dependent Utility

Suppose a total preorder \( \preceq \) on \( \mathcal{Z} \) is continuous, monotonic, and satisfies the dual independence axiom. Then it is risk-averse if and only if it has the integral representation (*) with a nondecreasing and convex function \( w : [0, 1] \to [0, 1] \) such that \( w(0) = 0 \) and \( w(1) = 1 \).
Two Objectives

- Minimize the expected outcome, the mean $\mathbb{E}[Z_x]$
- Minimize a scalar measure of uncertainty of $Z_x$, the risk $r[Z_x]$

$$r[Z] = \text{Var}[Z]$$  
(Markowitz’ model)

$$\sigma^+_\rho[Z] = (\mathbb{E}[\{(Z - \mathbb{E}[Z])^\rho\}^+])^{1/\rho}$$  
(semideviation)

$$\delta^+_\alpha[Z] = \min_\eta \mathbb{E}\left[\max\left(\eta - Z, \frac{\alpha}{1 - \alpha}(Z - \eta)\right)\right]$$  
(deviation from quantile)

$r[Z_x]$ is nonlinear w.r.t. probability and possibly nonconvex in $x$
Example: Portfolio Optimization

\( R_1, R_2, \ldots, R_n \) - random return rates of securities
\( x_1, x_2, \ldots, x_n \) - fractions of the capital invested in the securities

Return rate of the portfolio (negative of)

\[
Z_x = -\left( R_1 x_1 + R_2 x_2 + \cdots + R_n x_n \right)
\]

Risk Optimization with Fixed Mean

\[
\min_{x} \ r\left[Z_x\right] \\
s.t. \ \mathbb{E} \left[Z_x\right] = \mu \quad \text{(parameter)} \\
x \in X_0.
\]

Combined Mean–Risk Optimization

\[
\min_{x \in X_0} \ \rho \left[Z_x\right] = \mathbb{E} \left[Z_x\right] + \kappa r\left[Z_x\right], \quad 0 \leq \kappa \leq \kappa_{max}
\]

Interesting applications of parametric optimization
Suppose $Z$ has finitely many realizations $z_1, z_2, \ldots, z_S$ with probabilities $p_1, p_2, \ldots, p_S$

$$\rho(Z) = \mathbb{E}[Z] + \kappa \sigma_m^+[Z] = \mathbb{E}[Z] + \kappa \left( \mathbb{E}[(Z - \mathbb{E}Z)^m]_+ \right)^{1/m}$$

$$= \sum_{s=1}^S p_s z_s + \kappa \left( \sum_{s=1}^S p_s \left( z_s - \sum_{j=1}^S p_j z_j \right)_+^m \right)^{1/m}$$

**Equivalent Problem (for $m = 1$ - linear programming)**

$$\rho(Z) = \min_{v, \mu} \mu + \kappa \left( \sum_{s=1}^S p_s v_s^m \right)^{1/m}$$

s.t. 

$$\mu = \sum_{s=1}^S p_s z_s$$

$$v_s \geq z_s - \mu, \quad s = 1, \ldots, S$$

$$v_s \geq 0, \quad s = 1, \ldots, S$$
Suppose the vector of return rates has $S$ realizations with probabilities $p_1, p_2, \ldots, p_S$

$R_{js}$ - return rate of asset $j = 1, \ldots, n$ in scenario $s = 1, \ldots, S$

**Equivalent Problem (for $m = 1$ - linear programming)**

$$
\min_{x, z, v, \mu} \quad \mu + \kappa \left( \sum_{s=1}^{S} v_s^m \right)^{1/m} \\
\text{s.t.} \quad \mu = \sum_{s=1}^{S} p_s z_s \\
\quad z_s = - \sum_{j=1}^{n} R_{sj} x_j, \quad s = 1, \ldots, S \\
\quad v_s \geq z_s - \mu, \quad s = 1, \ldots, S \\
\quad v_s \geq 0, \quad s = 1, \ldots, S \\
\quad x \in X_0
$$
Basket of 719 Securities. Mean–Semideviation Model
Key Requirement: Monotonicity

\[ \rho(Z) = \mathbb{E}[Z] + \kappa r[Z] \]

Consistency with Stochastic Dominance (Ogryczak–R., 1997)

\[ \mathbb{E}[u(Z)] \leq \mathbb{E}[u(W)], \ \forall \text{ nondecreasing and convex } u(\cdot) \Rightarrow \rho[Z] \leq \rho[W] \]

Consistency with Pointwise Order (Artzner et. al., 1999)

\[ Z \leq W \text{ a.s. } \Rightarrow \rho[Z] \leq \rho[W] \]

Mean–semideviation and mean–deviation from quantile models are consistent for \( 0 \leq \kappa \leq 1 \), but not mean–variance.

Unique optimal solutions of consistent optimization models

\[ \min_{x \in X} \rho(Z_x) \]

cannot be strictly dominated (in the corresponding sense)
Coherent Risk Measures

Space of uncertain outcomes \( \mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P), \ p \in [1, \infty] \)

A functional \( \rho : \mathcal{Z} \to \overline{\mathbb{R}} \) is a coherent risk measure if it satisfies the following axioms

- **Convexity:** \( \rho(\lambda Z + (1 - \lambda)W) \leq \lambda \rho(Z) + (1 - \lambda)\rho(W) \)
  \( \forall \ \lambda \in (0, 1), \ Z, W \in \mathcal{Z} \)
- **Monotonicity:** If \( Z \leq W \) then \( \rho(Z) \leq \rho(W) \), \( \forall \ Z, W \in \mathcal{Z} \)
- **Translation Equivariance:** \( \rho(Z + a) = \rho(Z) + a \), \( \forall \ Z \in \mathcal{Z}, \ a \in \mathbb{R} \)
- **Positive Homogeneity:** \( \rho(\tau Z) = \tau \rho(Z) \), \( \forall \ Z \in \mathcal{Z}, \ \tau \geq 0 \)

Kijima-Ohnishi (1993) – no monotonicity
Artzner-Delbaen-Eber-Heath (1999–) - space \( \mathcal{L}_\infty \)
R.-Shapiro (2005) – spaces \( \mathcal{L}_p, \ldots \)

**Good news:** \( \mathbb{E}[Z] \) is coherent
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**Good news:** $E[Z]$ is coherent
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- **Monotonicity:** If $Z \leq W$ then $\rho(Z) \leq \rho(W)$, \[ \forall Z, W \in \mathcal{Z} \]

- **Translation Equivariance:** $\rho(Z + a) = \rho(Z) + a$, \[ \forall Z \in \mathcal{Z}, a \in \mathbb{R} \]

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Coherent Risk Measures

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- **Monotonicity:** If $Z \leq W$ then $\rho(Z) \leq \rho(W)$, $\forall Z, W \in \mathcal{Z}$

- **Translation Equivariance:** $\rho(Z + a) = \rho(Z) + a$, $\forall Z \in \mathcal{Z}, a \in \mathbb{R}$

- **Positive Homogeneity:** $\rho(\tau Z) = \tau \rho(Z)$, $\forall Z \in \mathcal{Z}, \tau \geq 0$

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**Good news:** $\mathbb{E}[Z]$ is coherent
Coherent Risk Measures

Space of uncertain outcomes \( \mathcal{Z} = L_p(\Omega, \mathcal{F}, P), \ p \in [1, \infty] \)

A functional \( \rho : \mathcal{Z} \to \overline{\mathbb{R}} \) is a **coherent risk measure** if it satisfies the following axioms

- **Convexity**: \( \rho(\lambda Z + (1 - \lambda) W) \leq \lambda \rho(Z) + (1 - \lambda) \rho(W) \)
  \( \forall \ \lambda \in (0, 1), \ Z, W \in \mathcal{Z} \)

- **Monotonicity**: If \( Z \leq W \) then \( \rho(Z) \leq \rho(W), \ \forall Z, W \in \mathcal{Z} \)

- **Translation Equivariance**: \( \rho(Z + a) = \rho(Z) + a, \ \forall Z \in \mathcal{Z}, a \in \mathbb{R} \)

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**Good news:** \( \mathbb{E}[Z] \) is coherent
Coherence of Mean–Semideviation

For simplicity, semideviation of order \( m = 1 \) with \( \kappa = 1 \):

\[
\rho(Z) = \mathbb{E}[Z] + \mathbb{E}[(Z - \mathbb{E}Z)_+] = \mathbb{E}\left\{ \max(\mathbb{E}[Z], Z) \right\}
\]

Convexity follows from the convexity of \( Z \mapsto \max(\mathbb{E}[Z], Z) \) a.s.

Monotonicity follows from monotonicity of \( Z \mapsto \max(\mathbb{E}[Z], Z) \) a.s.

Translation follows from translation of \( Z \mapsto \max(\mathbb{E}[Z], Z) \) a.s.

Pos. Homogeneity follows from pos. homogeneity of \( \max(\mathbb{E}[Z], Z) \) a.s.

Convex combination of coherent measures of risk is coherent

\[
\rho(Z) = \lambda_1 \rho_1(Z) + \lambda_2 \rho_2(Z) + \cdots + \lambda_L \rho_L(Z) \\
\lambda_1 + \lambda_2 + \cdots + \lambda_L = 1, \\
\lambda_1 \geq 0, \lambda_2 \geq 0, \ldots, \lambda_L \geq 0
\]

\[
\rho(Z) = \mathbb{E}[Z] + \kappa \mathbb{E}[(Z - \mathbb{E}Z)_+] \text{ is coherent for } \kappa \in [0, 1]
\]
The Value at Risk at level $\alpha \in (0, 1)$ of a random cost $Z \in \mathcal{Z}$:

$$V@R_{\alpha}^+(Z) \triangleq \inf \{ \eta : F_Z(\eta) \geq 1 - \alpha \} = F_Z^{-1}(1 - \alpha)$$

**Monotonicity:** $Z \leq V \implies V@R_{\alpha}^+(Z) \leq V@R_{\alpha}^+(V)$

**Translation:** $V@R_{\alpha}^+(Z + c) = V@R_{\alpha}^+(Z) + c$, for all $c \in \mathbb{R}$

**Positive Homogeneity:** $V@R_{\alpha}^+(\gamma Z) = \gamma V@R_{\alpha}^+(Z)$, for all $\gamma \geq 0$

However, it is not convex

**Counterexample:** Two independent variables

$$Z = \begin{cases} 0 \text{ with probability } 1 - p, \\ 1 \text{ with probability } p \end{cases} \quad V = \begin{cases} 0 \text{ with probability } 1 - p, \\ 1 \text{ with probability } p \end{cases}$$

For $p < \alpha < 1$ we have $V@R_{\alpha}^+(Z) = V@R_{\alpha}^+(V) = 0$

If $p < \alpha < 1 - (1 - p)^2$, we have non-convexity

$$V@R_{\alpha}^+(\lambda Z + (1 - \lambda) V) > 0 = \lambda V@R_{\alpha}^+(Z) + (1 - \lambda) V@R_{\alpha}^+(V)$$
Average Value at Risk

\[ \text{AV@R}_\alpha^+(Z) \triangleq \frac{1}{\alpha} \int_0^\alpha \text{V@R}_\beta^+(Z) \, d\beta \]

If the \((1 - \alpha)\)-quantile of \(Z\) is unique

\[ \text{AV@R}_\alpha^+(Z) = \frac{1}{\alpha} \int_{\text{V@R}_\alpha^+(Z)}^\infty z \, dF_Z(z) = \mathbb{E}[Z \mid Z \geq \text{V@R}_\alpha^+(Z)] \]

Extremal representation

\[ \text{AV@R}_\alpha^+(Z) = \inf_{\eta \in \mathbb{R}} \left\{ \eta + \frac{1}{\alpha} \mathbb{E}[(Z - \eta)_+] \right\} \]

The minimizer \(\eta = \text{V@R}_\alpha(Z)\)

Connection to weighted deviation from \(\alpha\)-quantile:

\[ \delta_\alpha^+(Z) = \text{AV@R}_\alpha^+(Z) - \mathbb{E}[Z], \quad \alpha \in [0, 1]. \]
Extremal representation

\[ \text{AV@R}_\alpha^+(Z) = \inf_{\eta \in \mathbb{R}} \left\{ \eta + \frac{1}{\alpha} \mathbb{E}\left[(Z - \eta)_+\right] \right\} \]

Convexity follows from joint convexity in \((\eta, Z)\) of \{\cdots\}

Monotonicity follows from monotonicity w.r.t. \(Z\) of \{\cdots\}

Translation follows from \(\eta \leftrightarrow \eta - c\) in \{\cdots\}

Pos. Homogeneity follows from pos. homogeneity in \((\eta, Z)\) of \{\cdots\}
Suppose $Z$ has finitely many realizations $z_1, z_2, \ldots, z_S$ with probabilities $p_1, p_2, \ldots, p_S$

\[
\begin{align*}
\min_{\nu, \eta} & \quad \eta + \frac{1}{\alpha} \sum_{s=1}^{S} p_s \nu_s \\
\text{s.t.} & \quad \nu_s \geq z_s - \eta, \quad s = 1, \ldots, S \\
& \quad \nu_s \geq 0, \quad s = 1, \ldots, S
\end{align*}
\]

For portfolios we have to add the constraints

\[
z_s = -\sum_{j=1}^{n} R_{sj} x_j, \quad s = 1, \ldots, S
\]

\[
x \in X_0
\]

and include $z$ and $x$ into the decision variables
Pairing of a linear topological space \( \mathcal{Z} \) with a linear topological space \( \mathcal{Y} \) of regular signed measures on \( \Omega \) with the bilinear form

\[
\langle \mu, Z \rangle = \mathbb{E}_\mu[Z] = \int_{\Omega} Z(\omega) \mu(d\omega)
\]

We assume standard conditions on pairing and the polarity: \( (\mathcal{Z}_+)^\circ = \mathcal{Y}_- \)

**Dual Representation Theorem**

If \( \rho : \mathcal{Z} \rightarrow \overline{\mathbb{R}} \) is a lower semicontinuous* coherent risk measure, then

\[
\rho(Z) = \max_{\mu \in \mathcal{A}} \int_{\Omega} Z(\omega) \mu(d\omega), \quad \forall Z \in \mathcal{Z}
\]

with a convex closed \( \mathcal{A} \subset \mathcal{P} \) (set of probability measures in \( \mathcal{Y} \)).


* Lower semicontinuity is automatic if \( \rho \) is finite and \( \mathcal{Z} \) is a Banach lattice.
Universality of AV@R

$Z \sim V$ means that $Z$ and $V$ have the same distribution, $\mu_Z = \mu_V$.

$\rho : \mathcal{Z} \rightarrow \mathbb{R}$ is law invariant if $Z \sim V \implies \rho(Z) = \rho(V)$

Kusuoka Theorem

If $(\Omega, \mathcal{F}, P)$ is atomless and $\rho : L_1(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ is law invariant, then

$$\rho(Z) = \sup_{m \in \mathcal{M}} \int_0^1 AV@R^+_{\alpha}(Z) \ m(d\alpha)$$

where $\mathcal{M}$ is a convex set of probability measures on $(0, 1]$.

Spectral measure

$$\rho(V) = \int_0^1 AV@R^+_{\alpha}(Z) \ m(d\alpha)$$

Spectral measures have dual utility form:

$$\rho(Z) = \int_0^1 F_Z^{-1}(\beta) \ dw(\beta)$$
“Minimize” over $x \in X$ a random outcome $Z_x(\omega) = f(x, \omega), \omega \in \Omega$

**Composite Optimization Problem**

$$\min_{x \in X} \rho(Z_x)$$ (P)

**Theorem**

Let $x \mapsto Z_x(\omega)$ be convex and $\rho(\cdot)$ be coherent. Suppose that $\hat{x} \in X$ is an optimal solution of (P) and $\rho(\cdot)$ is continuous at $Z_{\hat{x}}$. Then there exists a probability measure $\hat{\mu} \in \partial \rho(Z_{\hat{x}}) \subseteq \mathcal{A}$ such that $\hat{x}$ solves

$$\min_{x \in X} E_{\hat{\mu}}[Z_x] = \min_{x \in X} \max_{\mu \in \mathcal{A}} E_{\mu}[Z_x]$$

We also have the **duality relation**: 

$$\min_{x \in X} \rho(Z_x) = \max \inf_{\mu \in \mathcal{A}} E_{\mu}[Z_x]$$
Duality in Portfolio Optimization - Game Model

Suppose the vector of return rates of assets has $S$ realizations.

- $R_{js}$ - return rate of asset $j = 1, \ldots, n$ in scenario $s = 1, \ldots, S$

Portfolio return (negative) in scenario $s$

$$Z_s(x) = - \sum_{j=1}^{n} R_{js} x_j$$

Portfolio Problem

$$\min_{x \in X} \rho \left( Z(x) \right)$$

By homogeneity, we may assume that $\sum_{j=1}^{n} x_j = 1$

**Equivalent Matrix Game**

$$\max_{x \in X} \min_{\mu \in \mathcal{A}} \sum_{j=1}^{n} \sum_{s=1}^{S} x_j R_{js} \mu_s$$

- $x$ - mixed strategy of the investor
- $\mu$ - mixed strategy of the opponent (market)
Two-Stage Model

Expected-Value Model

$$\min_{x \in X} c^T x + \mathbb{E}[Q(x)]$$

where $Q(x)$ is the optimal value of the random second-stage problem

$$\min q^T y$$

s.t. $Tx + Wy = h,$

$$y \geq 0,$$

$(q, T, h)$ - random data of the second-stage problem

- $c$ is deterministic
- $(q, T, h)$ become known after the first stage

For finite scenario case - powerful decomposition methods
Two-Stage Model: Risk-Averse Version

\[
\min_{x \in X} \rho_1 \left( c^T x + Q(x) \right)
\]

where \( Q(x) \) is the optimal value of the second-stage problem

\[
Q(x) = \min_{\rho_2} \left( q^T y \right)
\]  
\[\text{s.t. } Tx + Wy = h, \]
\[y \geq 0,
\]

and \( \xi = (q, T, h) \) - random data of the second-stage problem

- \( c \) is random
- \( (T, h) \) become known after the first stage
- \( q \) may be still unknown (conditional distribution)
Second-stage scenarios: $c_s, T_s, h_s, s = 1, \ldots, S$
Final scenarios: $q_{sj}, j \in J(s)$

$$\min_{x \in X} \rho_1 \left( c^T x + Q(x) \right)$$

where $Q(x)$ is the optimal value of the second-stage problem; In scenario $s$ its value is

$$Q_s(x) = \min \rho_{2s} \left( q_s^T y \right)$$

s.t. $T_s x + W y = h_s,$

$$y \geq 0,$$

$q_s$ is random and has realizations $q_{sj}, j \in J(s)$

This structure of the problem follows from the general theory of dynamic measures of risk (lecture tomorrow)
Dual Representation of the Two-Stage Problem

Risk-averse first-stage problem

\[
\min_{x \in X} \max_{\mu \in A} \sum_{s=1}^{S} \mu_s \left[ c_s^T x + Q_s(x) \right]
\]

Risk-averse second-stage problem

\[
Q_s(x) = \min_y \max_{\nu \in B_s} \sum_{j \in J(s)} \nu_j q_j^T y \\
\text{s.t. } T_s x + W_s y = h_s \quad \text{(multipliers } \pi_s) \\
y \geq 0
\]

The sets of probability measures:

\[
A = \partial \rho_1(0) \\
B_s = \partial \rho_{2s}(0)
\]
Stochastic Dominance Constraints (Dentcheva–R., 2003–)

\( Z_x \) - random outcome (e.g., cost)

\( Y \) - benchmark random outcome, e.g. \( Y(\omega) = Z_{\bar{x}}(\omega) \) for some \( \bar{x} \in X \)

**New Model**

\[
\begin{align*}
\min & \quad \mathbb{E}[Z_x] \\
\text{subject to} & \quad Z_x \preceq_u Y \\
& \quad x \in X
\end{align*}
\]

(or some other objective)

(stochastic ordering constraint)

\( Z_x \) is preferred over \( Y \) by all decision makers having disutility functions in the generator \( \mathcal{U} \):

\[
\mathbb{E}[u(Z_x)] \leq \mathbb{E}[u(Y)] \quad \forall \ u \in \mathcal{U}
\]

All nondecreasing \( u(\cdot) \) - first order stochastic dominance \( \preceq_{st} \)

All nondecreasing convex \( u(\cdot) \) - increasing convex order \( \preceq_{icx} \)
Dominance Constrained Optimization

\[ \min \mathbb{E}[Z_x] \]
subject to \( Z_x \preceq_{icx} Y \)
\( x \in X \)

\( X \) - convex set in \( X \) (separable locally convex Hausdorff vector space)

\( x \mapsto Z_x \) is a continuous operator from \( X \) to \( L_1(\Omega, \mathcal{F}, P) \)

\( x \mapsto Z_x(\omega) \) is convex for \( P \)-almost all \( \omega \in \Omega \)

**Primal:** \( \mathbb{E}[u(Z_x)] \leq \mathbb{E}[u(Y)] \) for all convex nondecreasing \( u : \mathbb{R} \rightarrow \mathbb{R} \)

**Inverse:** \( \int_0^1 F^{-1}_{Z_x}(p) \, dw(p) \leq \int_0^1 F^{-1}_Y(p) \, dw(p) \) for all convex nondecreasing \( w : [0, 1] \rightarrow \mathbb{R} \)

**Main Results**

- Utility functions \( u : \mathbb{R} \rightarrow \mathbb{R} \) and rank dependent utility functions \( w : [0, 1] \rightarrow \mathbb{R} \) play the roles of Lagrange multipliers

- Expected utility models and rank dependent utility models are Lagrangian relaxations of the problem
Implied Utility Function

**Lagrangian in Direct Form**

\[
L(x, u) = \mathbb{E}[Z_x + u(Z_x) - u(Y)]
\]

\(u(\cdot)\) - convex function on \(\mathbb{R}\)

**Theorem**

Assume Uniform Dominance Condition (a form of Slater constraint qualification). If \(\hat{x}\) is an optimal solution of the problem then there exists a function \(\hat{u} \in \mathcal{U}\) such that

\[
L(\hat{x}, \hat{u}) = \min_{x \in X} L(x, \hat{u})
\]

(1)

\[
\mathbb{E}[^{\hat{u}}(Z_{\hat{x}})] = \mathbb{E}[^{\hat{u}}(Y)]
\]

(2)

Conversely, if for some function \(\hat{u} \in \mathcal{U}\) an optimal solution \(\hat{x}\) of (1) satisfies the dominance constraint and (2), then \(\hat{x}\) is optimal.
Implied Rank Utility (Distortion) Function

**Lagrangian in Inverse Form**

\[ \Phi(x, w) = \int_0^1 F_{Z_x}^{-1}(p) \, d(p + w(p)) - \int_0^1 F_Y^{-1}(p) \, dw(p) \]

\( w(\cdot) \) - convex function on [0, 1]

**Theorem**

Assume Uniform Dominance Condition (a form of Slater constraint qualification). If \( \hat{x} \) is an optimal solution of the problem, then there exists a function \( \hat{w} \in \mathcal{W} \) such that

\[ \Phi(\hat{x}, \hat{w}) = \min_{x \in \mathcal{X}} \Phi(x, \hat{w}) \quad (3) \]

\[ \int_0^1 F_{Z_{\hat{x}}}^{-1}(p) \, d\hat{w}(p) = \int_0^1 F_Y^{-1}(p) \, d\hat{w}(p) \quad (4) \]

If for some \( \hat{w} \in \mathcal{W} \) an optimal solution \( \hat{x} \) of (3) satisfies the inverse dominance constraint and (4), then \( \hat{x} \) is optimal