Dealing with Uncertainty

in Decision Making Models

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I. A product mix problem
A formulation

A furniture manufacturer must choose $x_j \geq 0$, how many dressers of type $j = 1, \ldots, 4$ to manufacture so as to maximize profit

$$\sum_{j=1}^{4} c_j x_j = 12x_1 + 25x_2 + 21x_3 + 40x_4$$

The constraints:

$$t_{c1}x_1 + t_{c2}x_2 + t_{c3}x_3 + t_{c4}x_4 \leq d_c$$
$$t_{f1}x_1 + t_{f2}x_1 + t_{f3}x_1 + t_{f4}x_1 \leq d_f$$

$t_{cj}$ ($t_{fj}$) = carpentry (finishing) man-hours: dresser type $j$

$d_c$ ($d_f$) = total time available for carpentry (finishing)
Product mix problem (2)

Solution via linear programming:

$$\max \langle c, x \rangle \text{ so that } Tx \leq d, \ x \in \mathbb{R}^n_+.$$ 

With

$$T = \begin{bmatrix} t_{c1} & t_{c2} & t_{c3} & t_{c4} \\ t_{f1} & t_{f2} & t_{f3} & t_{f4} \end{bmatrix} = \begin{bmatrix} 4 & 9 & 7 & 10 \\ 1 & 1 & 3 & 40 \end{bmatrix}, \quad \begin{bmatrix} d_c \\ d_f \end{bmatrix} = \begin{bmatrix} 6000 \\ 4000 \end{bmatrix}$$

Optimal: $$x^d = \left(\frac{4000}{3}, 0, 0, \frac{200}{3}\right)$$

Value: $18,667.$
Product mix problem (3)

But . . . “reality” can’t be ignored!

\[ t_{cj} = t_{cj} + \eta_{cj}, \quad t_{fj} = t_{fj} + \eta_{fj} \]

<table>
<thead>
<tr>
<th>entry</th>
<th>possible values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_c + \zeta_c ):</td>
<td>5,873 5,967 6,033 6,127</td>
</tr>
<tr>
<td>( d_f + \zeta_f ):</td>
<td>3,936 3,984 4,016 4,064</td>
</tr>
</tbody>
</table>

10 random variables, say, 4 possible values each

\[ L = 1,048,576 \text{ possible pairs } (T^l, d^l) \]
Product mix problem (4)

What if \( \sum_{j=1}^{4} (t_{cj} + \eta_{cj}) x_j > d_c + \zeta_c \)? \( \implies \) overtime

With \( \xi = (\eta_{\cdot\cdot}, \zeta_{\cdot\cdot}) \), **recourse**: \( (y_c(\xi), y_f(\xi)) \) @ cost \( (q_c, q_f) \).

\[
\begin{align*}
\max & \quad \langle c, x \rangle - p_1 \langle q, y^1 \rangle - p_2 \langle q, y^2 \rangle \cdots - p_L \langle q, y^L \rangle \\
\text{s.t.} & \quad T^1 x - y^1 \leq d^1 \\
& \quad T^2 x - y^2 \leq d^2 \\
& \quad \vdots \\
& \quad T^L x - y^L \leq d^L \\
& \quad x \geq 0, \quad y^1 \geq 0, \quad y^2 \geq 0, \quad \cdots \quad y^L \geq 0.
\end{align*}
\]

Structured large scale l.p. \( (L \approx 10^6) \)
Define $\Xi = \{\xi = (\eta, \zeta)\}$, $p_\xi = \text{prob} [\xi = \xi]$

$$Q(\xi, x) = \max \{\langle -q, y \rangle \mid T_\xi x - y \leq d_\xi, y \geq 0\}$$

$$EQ(x) = E\{Q(\xi, x)\} = \sum_{\xi \in \Xi} p_\xi Q(\xi, x)$$

the equivalent deterministic program (DEP):

$$\max \langle c, x \rangle + EQ(x) \text{ so that } x \in \mathbb{R}_+^n.$$ 

a non-smooth convex optimization problem: $EQ$ concave.
Solution of DEP, or large scale l.p.,:

Optimal: \( x^* = (257, 0, 665.2, 33.8) \)

expected Profit: $18,051

The solution \( x^* \) is robust: it considered all \( \approx 10^6 \) possibilities.
Solution of DEP, or large scale l.p.,:

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expected Profit: $18,051

The solution \( x^* \) is \textit{robust}: it considered all \( \approx 10^6 \) possibilities.

Recall: \( x^d = (1333.33, 0, 0, 66.67) \)

expected “profit” relying on \( x^d = $16,942. \)
Product mix problem (6)

Solution of DEP, or large scale l.p.,:

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Recall: \(x^d = (1333.33, 0, 0, 66.67)\)

expected “profit” relying on \(x^d = $16,942\).

- \(x^d\) is not close to optimal
- \(x^d\) isn’t pointing in the right direction
Stochastic Programming relies on:

- linear, non-linear, mixed-integer programming
- large scale: decomposition methods, structured programs, grid computing
- Variational Analysis: non-smooth, duality, epi-convergence (approximations), etc.
- Probability: stochastic processes, asymptotic laws
- Statistics: estimation, lack of data issues
- Functional Analysis, Combinatorial Geometry, etc.
II. Modeling, modeling & modeling!
Uncertain parameters

Deterministic Optimization problem:

$$\min f_0(x) \text{ so that } x \in S \subset \mathbb{R}^n$$
Uncertain parameters

Deterministic Optimization problem:

$$\min f_0(x) \text{ so that } x \in S \subset \mathbb{R}^n$$

Uncertain parameters: $$\xi \in \Xi \subset \mathbb{R}^N$$,

$$\min f_0(\xi, x) \text{ so that } x \in S(\xi) \subset \mathbb{R}^n$$
Deterministic Optimization problem:

\[
\min f_0(x) \text{ so that } x \in S \subset \mathbb{R}^n
\]

Uncertain parameters: \( \xi \in \Xi \subset \mathbb{R}^N \),

\[
\min f_0(\xi, x) \text{ so that } x \in S(\xi) \subset \mathbb{R}^n
\]

*Wait-and-see* solution ??

\[
x(\xi) \in \text{argmin} \left\{ f_0(\xi, x) \mid x \in S(\xi) \right\}
\]

What’s needed: a *here-and-now* solution.
The NewsVendor Problem

- $\xi \in \Xi \subset \mathbb{R}_+$ demand for a (perishable) good, e.g., plant capacity, overbooking, etc.
- $x \geq 0$ quantity ordered @ unit cost: $c = 10$
- $y \geq 0$ quantity sold, per unit profit $r = 15$

Total revenue (possibly negative):

$$-cx + (c + r)y \text{ where } 0 \leq x,$$

$$0 \leq y \leq \min \{x, \xi\}$$

Find optimal $x^*$!
The “deterministic” approach

Pick $\hat{\xi} \in \Xi$ (guessing the future) and solve

$$\min f_0(\hat{\xi}, x) \text{ so that } x \in S(\hat{\xi}) \subset \mathbb{R}^n$$
The “deterministic” approach

Pick $\hat{\xi} \in \Xi$ (guessing the future) and solve

$$\min f_0(\hat{\xi}, x) \text{ so that } x \in S(\hat{\xi}) \subset \mathbb{R}^n$$

**NewsVendor:** $\Xi = [0, 150]$, pick $\hat{\xi} = 75$,

$$\max -cx + (c + r)y$$

$$x \geq 0, \quad 0 \leq y \leq \min\{x, \hat{\xi}\}$$

Solution: $x^o = y^o = \hat{\xi}$, obj. value $= r\hat{\xi} = 1125$

But doesn’t tell much about “profit” if $\xi \neq 75$!
Scenario Analysis

Pick $\xi^1, \ldots, \xi^L$ (scenarios), and for each $\xi^l$ find:

$$x^l \in \text{argmin} \left\{ f_0(\xi^l, x) \mid x \in S(\xi^l) \right\}$$

and “reconcile” the solutions to obtain $x^o$. 
Pick $\xi^1, \ldots, \xi^L$ (scenarios), and for each $\xi^l$ find:

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and “reconcile” the solutions to obtain $x^o$.

**NewsVendor:** pick $\xi^1 = 10, \xi^2 = 20, \ldots, \xi^{15} = 150,$

$$(x^l, y^l) \in \arg\max \{-cx + (c + r)y \mid y \leq \min[\xi^l, x] \}$$

$\forall x \geq 0, y \geq 0$

**Wait-and-see sol’ns:** $x^l = \xi^l$. “Reconciliation”?

No help in choosing $x^o$ the quantity to order.
\( \xi: \) Estimated Density \( h \)

\( \xi \) log-normal: 

\[
h(z) = \left( z \tau \sqrt{2\pi} \right)^{-1} e^{-\frac{(\ln z - \theta)^2}{2\tau^2}}
\]

\( \theta = 4.43, \tau = 0.38; \quad H(z) = \int_0^z h(s) \, ds \)

from data, expert(s), all information available

might affect choice of \( \hat{\xi} \), scenarios: \( \xi^1, \ldots \)
Maximize Expected Return

$$\max -cx + E\{(c + r)y_\xi\}$$

so that $x \geq 0$, $0 \leq y_\xi \leq \min[\xi, x]$.

The *equivalent deterministic program*:

$$\max_{x \geq 0} -cx + EQ(x), \quad EQ(x) = E\{Q(\xi, x)\}$$

where $Q(\xi, x) = \begin{cases} (c + r)\xi & \text{if } \xi \leq x, \\ (c + r)x & \text{if } \xi \geq x \end{cases}$

$$EQ(x) = (c + r) \left( \int_0^x \xi H(d\xi) + \int_x^\infty x H(d\xi) \right)$$
Optimal: Expected Profit

\[ x^* = H^{-1} \left( \frac{r}{c + r} \right) = H^{-1}(0.6) = 99.2 \]

for \( c = 10, r = 15 \).
News Vendor’s Objective

- $E_w = 90$, $Stddev = 36$
- $Cost = 10$, $Revenue = 15$
- Opt. sol’n: 99.2, $E[profit] = 1,084$
- Sol’n guess: 75, $E[profit] = 1,000$
but is maximum expected return the “real” objective?
The Returns’ Densities
Choosing the Returns’ Distribution

Distribution Functions

- $x=72.2$
- $x=92.2$
- $x=112.2$
- $x=132.2$
- $x=152.2$
Decision Criteria

Reducing the choice of a distribution function to the choice of a “number”

- maximize expected return (scaled?),
- max. $E\{\text{return}\}$ & minimize customers lost,
- minimize Value-at-Risk (VaR),
- minimize the (buffered) probability of failure,
- minimizing a Measure
- variants & combinations of the above
Maximizing Expected Utility

“generic” stochastic optimization problem:
\[
\max E\{f_0(\xi, x)\} \text{ such that } f_i(\xi, x) \leq 0, \ i = 1, \ldots, m,
\]

Risk-averse or risk-seeking \(\implies\) utility function

von Neuman-Morgenstern: under “rationality” (axiomatics) there exists a utility function \(u\) such that \(\bar{x} \in \arg\max E\{u(f_0(\xi, x))\}\) (subject to the constraints) identifies the preferred return’s distribution

Modeling hurdle: no blueprint for \(u\)’s design!
Robust Optimization

“generic” optimization problem: \[ \max \gamma \]
so that \( \gamma - f_0(\xi, x) \leq 0, \]
f_i(\xi, x) \leq 0, \( i = 1, \ldots, m, \)

“robust” counterpart: \[ \max \gamma \]
so that \( \gamma - f_0(\xi, x) \leq 0, \forall \xi \in U \subset \Xi \)
f_i(\xi, x) \leq 0, \( i = 1, \ldots, m, \forall \xi \in U, \)

Challenges:
- formulate a computationally tractable robust counterpart
- specify reasonable uncertainty for set \( U \)
Reliability: Chance Constraints

Satisfy constraints with probability $\alpha \in (0, 1]$

$$\min f_0(x) \text{ so that } \text{prob. } [x \in S(\xi)] \geq \alpha$$

Variant:

$$\min f_0(x) \text{ so that } \text{prob. } [f_i(\xi, x) \leq 0] \geq \alpha_i, \ i \in I$$

$\alpha_i$ dictated by

- contractual obligations, company policy, guess, etc.

- often nonconvex; convex alternative, buffered prob.
Stochastic Dominance: $D_x(s) \leq D_{\hat{x}}(s), \quad \forall s$

\[\iff\] probability of the return to be $\leq s$

always smaller when choosing $x$ rather than $\hat{x}$

unfortunately unusual
Second order Stochastic Dominance

\[ D^2(s) = \int_{-\infty}^{s} D(\xi) \, d\xi = E\{(s - \xi)_+\} \]

\( D^2 \): the expected shortfall function
Stochastic Dominance Constraint

NewsVendor problem

\[
\max rx, \quad x \geq 0
\]

such that \( D^2_x(s) \leq G^2(s) \), \( s \in [\alpha, \beta] \)

given a “desirable” distribution function \( G \)

\( D_x \): distribution of actual return, decision \( x \)

\( \begin{align*}
rx \text{ when } \xi \leq x \text{ and } (c + r)\xi - cx \text{ when } \xi < x
\end{align*} \)

leads to a semi-infinite optimization problem used in portfolio optimization, for example
Perturbing the Probability Measure

stress testing via distribution contamination

Newsboy problem
Objective function
Ew = 90, Stdev=36
Cost=10, Revenue=15
Optimal sol’n: 99.2

102.4 Optimal sol’n
Ew = 100, Stdev= 40
"loss":  0.15%
A few references - Books


A few references - Tutorials/Surveys

1. Excellent tutorials at stoporg.org
2. Upcoming tutorial by A. Shapiro and previous tutorial by J. Luedtke
3. Overview of Risk-averse models:
   R.T. Rockafellar and J. Royset,
   Engineering Decisions under Risk-Averseness,